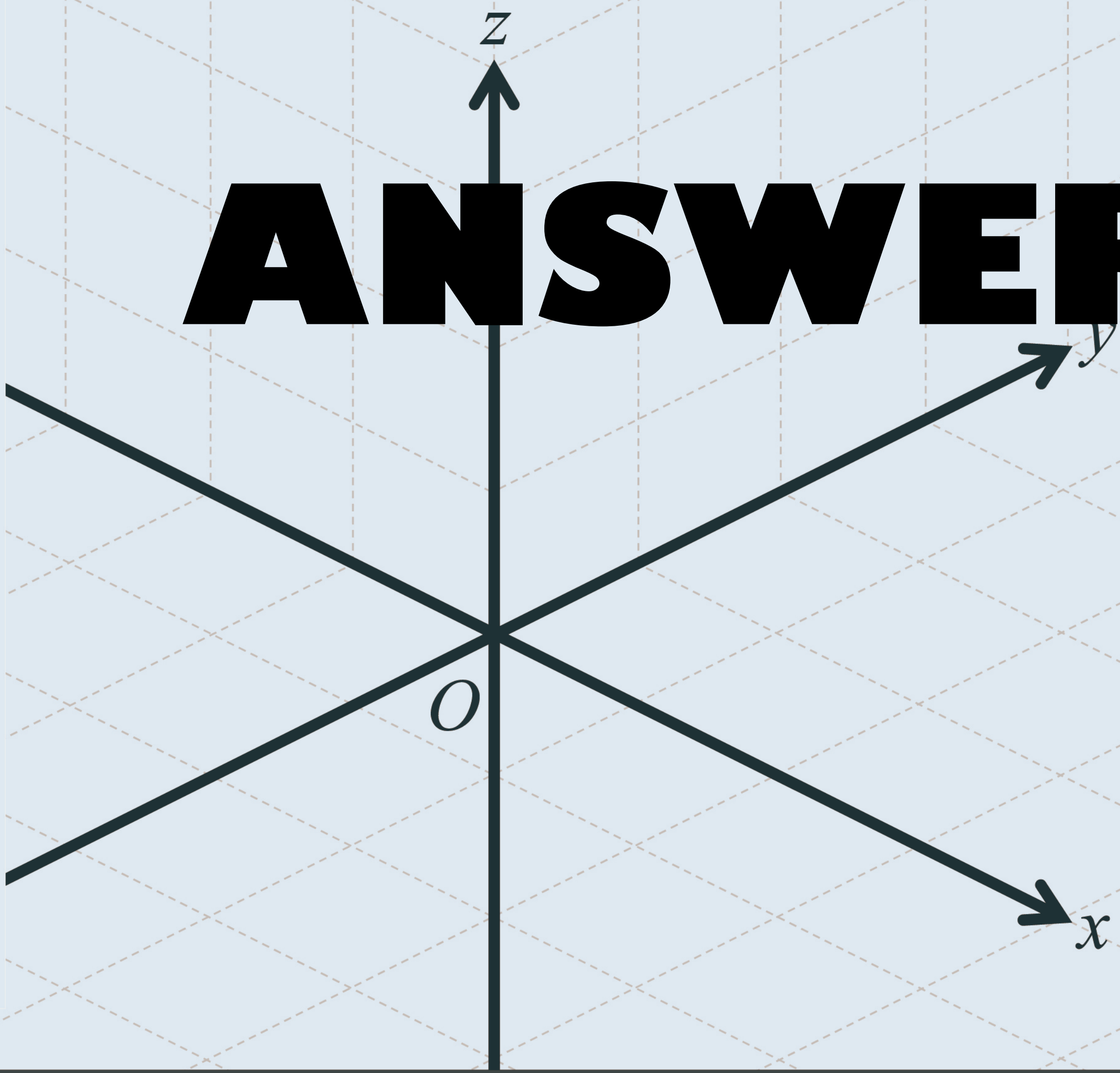
YOUR PRACTICE SET

ANALYSIS AND APPROACHES FOR IBDP MATHEMATICS

Book 2

ANSWERS



Stephen Lee
Michael Cheung
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- Compulsory Topics for MAA HL Students
- 80 Example Questions + 320 Intensive Exercise Questions
- Comprehensive Paper 3 Analysis and Practice Questions
- Holistic Exploration on Assessment-styled Questions

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Chapter 1 Solution

Exercise 1

1. (a) $|2x+1| \le 4x-1$

$-(4x-1) \le 2x+1 \le 4x-1$ M1

$-4x+1 \le 2x+1$ and $2x+1 \le 4x-1$

$0 \le 6x$ and $2 \le 2x$

$x \ge 0$ and $x \ge 1$ A1

$\therefore x \ge 1$ A1

[3]

(b) $|2x+1| > |4x-1|$

$|2x+1|^2 > |4x-1|^2$

$4x^2+4x+1 > 16x^2-8x+1$ M1

$12x^2-12x < 0$ (A1) for correct inequality

$12x(x-1) < 0$

$\therefore 0 < x < 1$ A2

[4]

2. (a) $\dfrac{|x|+1}{3} < 2|x| - 5$

$\dfrac{1}{3}|x| + \dfrac{1}{3} < 2|x| - 5$ M1

$\dfrac{16}{3} < \dfrac{5}{3}|x|$

$|x| > \dfrac{16}{5}$

$\therefore x > \dfrac{16}{5}$ or $x < -\dfrac{16}{5}$ A2

[3]

(b) $\left|\dfrac{x+1}{3}\right| < |2x-5|$

$\left|\dfrac{x+1}{3}\right|^2 < |2x-5|^2$

$\dfrac{x^2+2x+1}{9} > 4x^2 - 20x + 25$ M1

$x^2 + 2x + 1 > 36x^2 - 180x + 225$

$35x^2 - 182x + 224 < 0$ (A1) for correct inequality

$7(x-2)(5x-16) < 0$

$\therefore 2 < x < \dfrac{16}{5}$ A2

[4]

3. $f(|x|) > 5$

$2|x| + \dfrac{3}{|x|} > 5$ M1

$2|x|^2 + 3 > 5|x|$

$2|x|^2 - 5|x| + 3 > 0$ (A1) for correct inequality

$(|x|-1)(2|x|-3) > 0$

$\therefore |x| < 1$ or $|x| > \dfrac{3}{2}$ A1

$\therefore 0 < x < 1$ or $x > \dfrac{3}{2}$ A2

[5]

4. $f(|x|) \geq |x| + 2$

$\frac{|x|^3 - 14|x| + 8}{|x| + 4} \geq |x| + 2$ M1

$|x|^3 - 14|x| + 8 \geq (|x| + 2)(|x| + 4)$

$|x|^3 - 14|x| + 8 \geq |x|^2 + 6|x| + 8$

$|x|^3 - |x|^2 - 20|x| \geq 0$ (A1) for correct inequality

$|x|\left(|x|^2 - |x| - 20\right) \geq 0$

$|x|^2 - |x| - 20 \geq 0$ M1

$(|x| + 4)(|x| - 5) \geq 0$

$\therefore |x| < -4$ (*Rejected*) or $|x| > 5$ A1

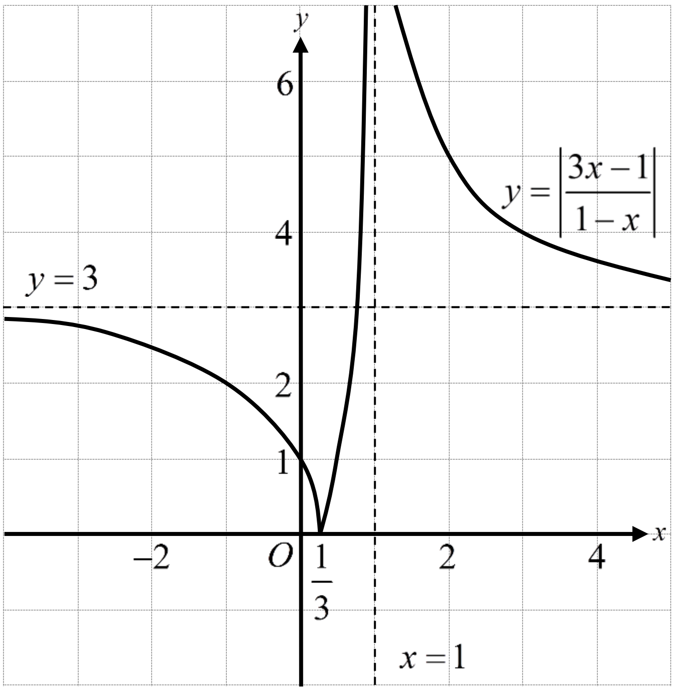
$x > 5$ (*Rejected*) or $x < -5$ A1

[5]

Exercise 2

1. (a) For correct asymptotes — A1
For correct intercepts — A1
For correct shape — A2

[4]



(b) $\left|\frac{3x-1}{1-x}\right|=5$

$\frac{3x-1}{1-x}=5$ or $\frac{3x-1}{1-x}=-5$ — M1

$3x-1=5(1-x)$ or $3x-1=-5(1-x)$

$3x-1=5-5x$ or $3x-1=-5+5x$

$8x=6$ or $-2x=-4$

$x=\frac{3}{4}$ or $x=2$ — A2

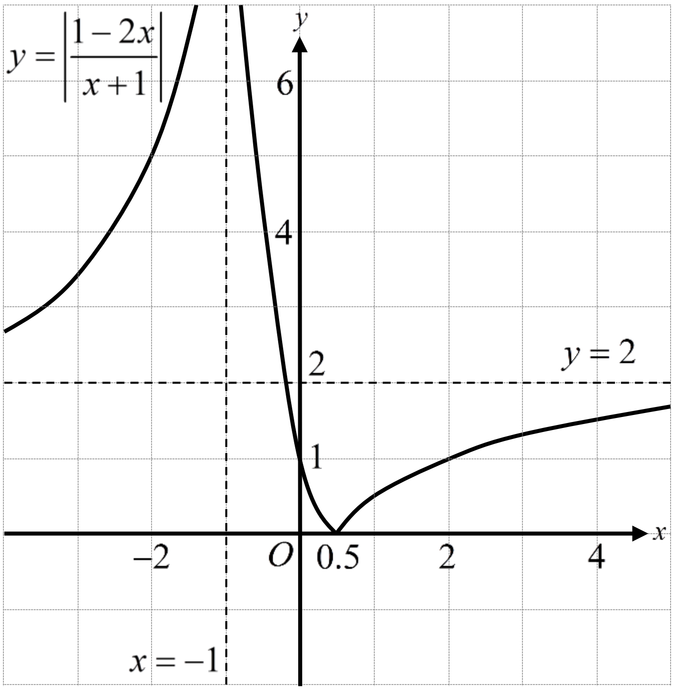
[3]

2. (a) For correct asymptotes A1

For correct intercepts A1

For correct shape A2

[4]



(b) 0 and 2 A2

[2]

3. (a) $f^{-1}(f(x)) = x$

$\therefore \dfrac{2f(x)}{af(x)+2} = x$ M1

$\dfrac{2\left(\dfrac{2x}{2-x}\right)}{a\left(\dfrac{2x}{2-x}\right)+2} = x$

$\dfrac{2(2x)}{a(2x)+2(2-x)} = x$ M1

$\dfrac{4x}{2ax+4-2x} = x$

$4x = 2ax^2 + 4x - 2x^2$

$2x^2 = 2ax^2$

$a = 1$ A1

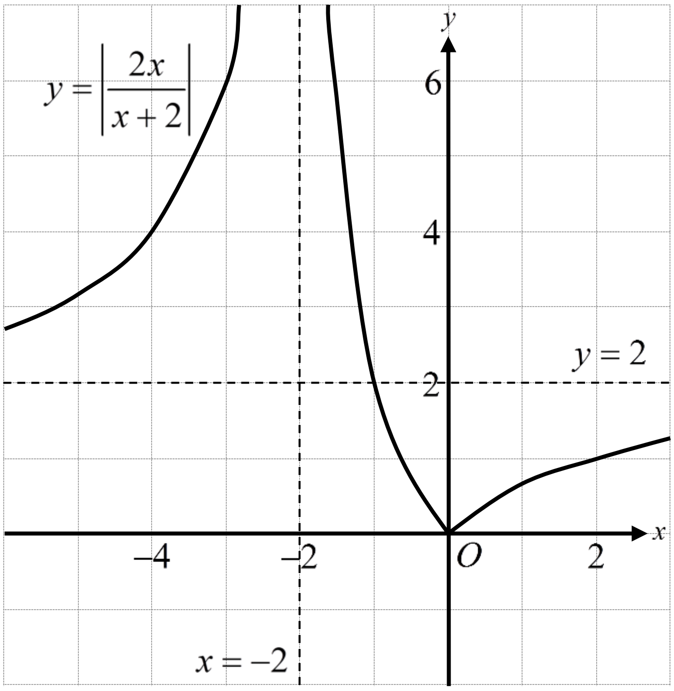
[3]

(b) For correct asymptotes A1

For correct intercept A1

For correct shape A2

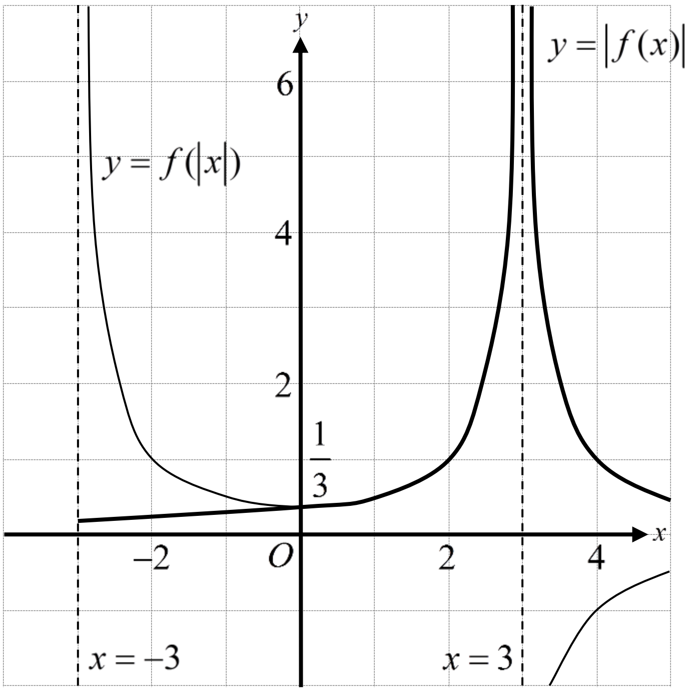
[4]



4. (a) (i) For correct asymptotes A1

For correct intercept A1

For correct shape A1



(ii) For correct asymptotes A1

For correct intercept A1

For correct shape A1

[6]

(b) 1 A1

[1]

Exercise 3

1. (a) $(f\circ g)(x)=\dfrac{4g(x)+1}{\sqrt{g(x)}}$

$(f\circ g)(x)=\dfrac{4(4x^2)+1}{\sqrt{4x^2}}$ M1

$(f\circ g)(x)=\dfrac{16x^2+1}{2x}$ A1

[2]

(b) $(f\circ g)(-x)=\dfrac{16(-x)^2+1}{2(-x)}$ M1

$(f\circ g)(-x)=-\dfrac{16x^2+1}{2x}$ A1

$(f\circ g)(-x)=-(f\circ g)(x)$

Thus, $(f\circ g)(x)$ is an odd function. AG

[2]

(c) $\{y:y\ge 4\}$ A1

[1]

2. (a) $f(-x)=(-x)^4-(-x)^2$ M1

$f(-x)=x^4-x^2$ A1

$f(-x)=f(x)$

Thus, $f(x)$ is an even function. AG

[2]

(b) As $f(x)$ is an even function, the minimum point of f has x-coordinate $\dfrac{1}{\sqrt{2}}$ for $x>0$.

Thus, the two minimum points of $f(x)$ have the same y-coordinate. (R1) for correct argument

$f\left(-\dfrac{1}{\sqrt{2}}\right)=\left(-\dfrac{1}{\sqrt{2}}\right)^4-\left(-\dfrac{1}{\sqrt{2}}\right)^2$ M1

$f\left(-\dfrac{1}{\sqrt{2}}\right)=\dfrac{1}{4}-\dfrac{1}{2}$

$f\left(-\dfrac{1}{\sqrt{2}}\right)=-\dfrac{1}{4}$

Thus, the range of $f(x)$ is $\left\{y:y\ge-\dfrac{1}{4}\right\}$. A1

[3]

3. (a) $f(-x)=\frac{-x}{(-x)^2+0.19}$ M1

$f(-x)=-\frac{x}{x^2+0.19}$ A1

$f(-x)=-f(x)$

Thus, $f(x)$ is an odd function. AG

[2]

(b) $f(a)=a$

$\frac{a}{a^2+0.19}=a$

$a=a(a^2+0.19)$ (M1) for valid approach

$a(a^2+0.19-1)=0$

$a(a^2-0.81)=0$

$a=0$ or $a^2=0.81$

$a=0$, $a=-0.9$ or $a=0.9$ A2

[3]

(c) $y=0$ A1

[1]

4. (a) $f(|-x|)=\frac{2-5|-x|}{2-9|-x|}$ M1

$f(|-x|)=\frac{2-5|x|}{2-9|x|}$ A1

$f(|-x|)=f(|x|)$

Thus, $f(|x|)$ is an even function. AG

[2]

(b) $y=\frac{5}{9}$ A1

[1]

(c) $x=\frac{2}{9}$, $x=-\frac{2}{9}$ A2

[2]

Exercise 4

1. (a) $\{x : x \geq 7\}$ A2

[2]

(b) $f(x) = 10 - 2(x - 7)$ (A1) for correct function

$y = -2x + 24$

$\Rightarrow x = -2y + 24$ (M1) for swapping variables

$2y = 24 - x$

$y = 12 - \frac{1}{2}x$

$\therefore f^{-1}(x) = 12 - \frac{1}{2}x$ A1

[3]

2. (a) $\{x : x \leq 2\}$ A2

[2]

(b) $f(x) = -(x^3 - 8)$ (A1) for correct function

$y = -x^3 + 8$

$\Rightarrow x = -y^3 + 8$ (M1) for swapping variables

$y^3 = 8 - x$

$y = \sqrt[3]{8 - x}$

$\therefore f^{-1}(x) = \sqrt[3]{8 - x}$ A1

[3]

(c) $\frac{1}{2}$ A1

[1]

3. (a) $\left\{x : -\frac{1}{2} \le x \le 4\right\}$ A2

[2]

(b) $f(x) = (2x+1)^2$

$y = (2x+1)^2$

$\Rightarrow x = (2y+1)^2$ (M1) for swapping variables

$\sqrt{x} = 2y+1$

$\sqrt{x} - 1 = 2y$

$y = \frac{\sqrt{x}-1}{2}$

$\therefore f^{-1}(x) = \frac{\sqrt{x}-1}{2}$ A1

[2]

(c) $(f \circ g)(x) = (4x+7)^2$

$g(x) = f^{-1}((4x+7)^2)$ M1

$g(x) = \frac{\sqrt{(4x+7)^2}-1}{2}$

$g(x) = \frac{4x+7-1}{2}$

$g(x) = \frac{4x+6}{2}$

$g(x) = 2x+3$ A1

[2]

4. (a) $\{x : -5 \le x \le 3\}$ A2

[2]

(b) $f(x) = -(x-3)^2 + 5$

$y = -(x-3)^2 + 5$

$\Rightarrow x = -(y-3)^2 + 5$ (M1) for swapping variables

$(y-3)^2 = 5 - x$

$y - 3 = \sqrt{5-x}$ (*Rejected*) or $y - 3 = -\sqrt{5-x}$ A1

$y = -\sqrt{5-x} + 3$

$\therefore f^{-1}(x) = -\sqrt{5-x} + 3$ A1

[3]

(c) $(g^{-1} \circ f^{-1})(x) = 2x$

$f^{-1}(x) = g(2x)$ M1

$g(2x) = -\sqrt{5-x} + 3$

$g\left(2\left(\frac{1}{2}x\right)\right) = -\sqrt{5 - \frac{1}{2}x} + 3$ A1

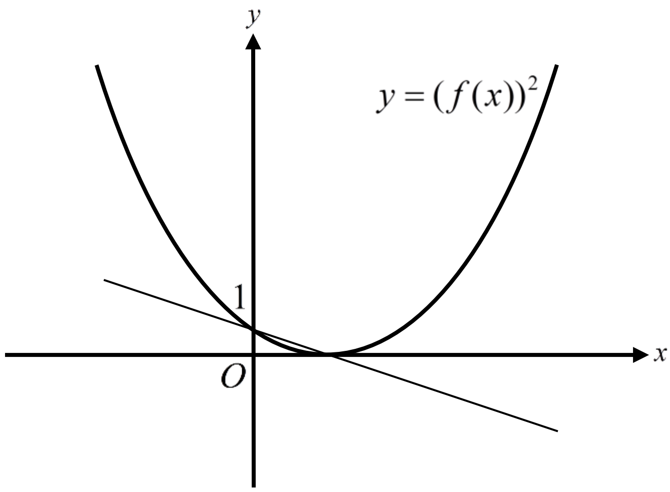
$\therefore g(x) = -\sqrt{5 - \frac{1}{2}x} + 3$ A1

[3]

Exercise 5

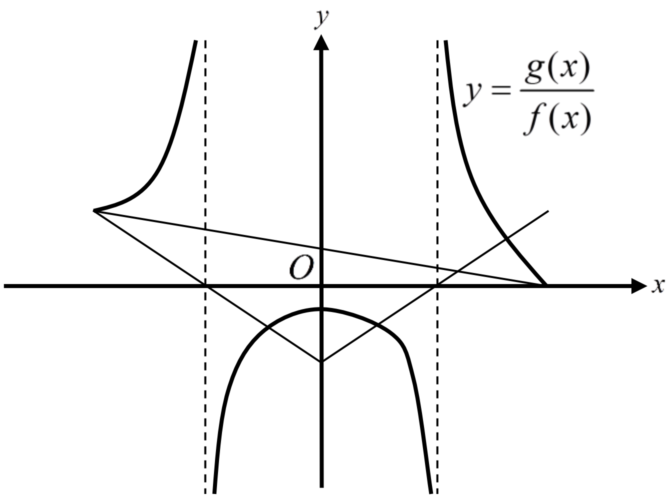
1. For correct intercepts A2
For correct concavity A1
For correct minimum point A1

[4]



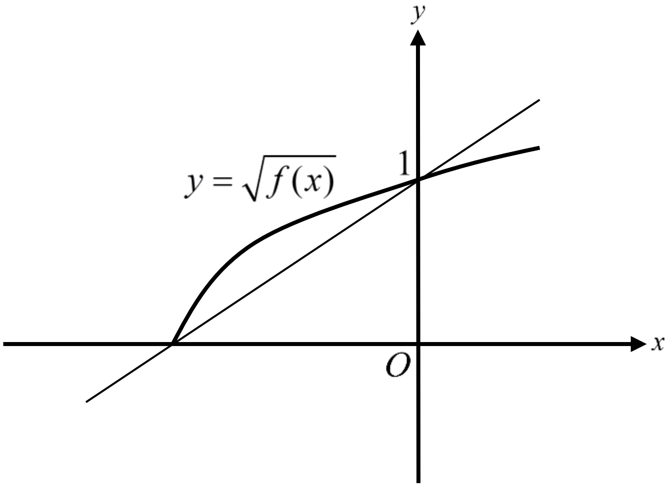
2. For correct intercepts A2
For correct asymptotes A2
For correct concavity A1

[5]



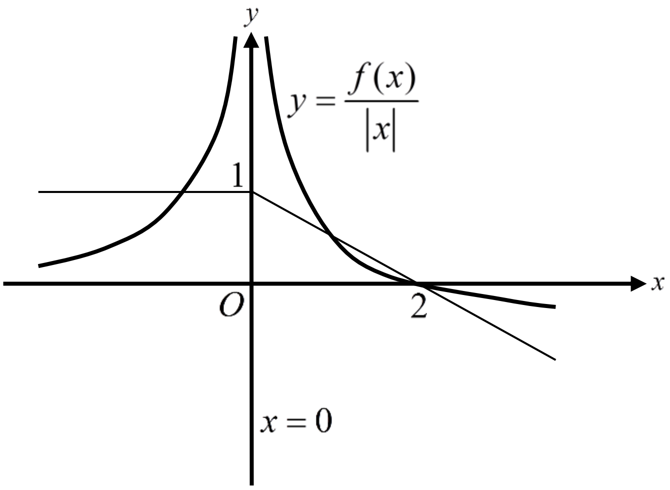
3. For correct intercepts A2
For correct concavity A1
For correct domain A1

[4]



4. For correct x-intercept A1
For correct asymptote A1
For correct concavity A2

[4]



Chapter 2 Solution

Exercise 6

1. $a(-2)^3-(-2)^2+b(-2)+3=-13$ (M1) for remainder theorem

$-8a-4-2b+3=-13$

$-8a+12=2b$

$b=6-4a$ A1

$a(3)^3-3^2+3b+3=27$

$27a-9+3b+3=27$

$\therefore 27a-9+3(6-4a)+3=27$ (M1) for substitution

$27a-9+18-12a+3=27$

$15a=15$

$a=1$ A1

$b=6-4(1)$

$b=2$ A1

[5]

2. $4(2)^3-2k(2)^2-5=3(2)^3-k^2(2)^2+15$ M1A1

$32-8k-5=24-4k^2+15$

$27-8k=24-4k^2+15$ (A1) for simplification

$4k^2-8k-12=0$ A1

$k^2-2k-3=0$

$(k+1)(k-3)=0$

$k+1=0$ or $k-3=0$

$k=-1$ or $k=3$ A2

[6]

3. (a) $4(2)^3+p(2)^2+q(2)=4(-2)^3+p(-2)^2+q(-2)$ M1A1

$32+4p+2q=-32+4p-2q$

$4q=-64$

$q=-16$ A1

[3]

(b) $p\in\mathbb{R}$ A1

[1]

4. $f(p) = q$

$p^4 + 4p^2 - 3p + 2 = q$ (M1) for remainder theorem

$f(-p) = q + 12$

$(-p)^4 + 4(-p)^2 - 3(-p) + 2 = q + 12$ (M1) for remainder theorem

$p^4 + 4p^2 + 3p + 2 = q + 12$

$\therefore p^4 + 4p^2 + 3p + 2 = p^4 + 4p^2 - 3p + 2 + 12$ (M1) for substitution

$3p + 2 = -3p + 14$

$6p = 12$

$p = 2$ A1

$q = 2^4 + 4(2)^2 - 3(2) + 2$

$q = 28$ A1

[5]

Exercise 7

1. $f(2)=0$

$a(2)^3+b(2)^2-13(2)+6=0$ (M1) for factor theorem

$8a+4b-26+6=0$

$4b=20-8a$

$b=5-2a$ A1

$f(-3)=0$

$a(-3)^3+b(-3)^2-13(-3)+6=0$

$-27a+9b+39+6=0$

$\therefore -27a+9(5-2a)+39+6=0$ (M1) for substitution

$-27a+45-18a+39+6=0$

$-45a=-90$

$a=2$ A1

$b=5-2(2)$

$b=1$ A1

[5]

2. $f(4)=0$

$4^3+p(4)^2+q(4)+48=0$ (M1) for factor theorem

$64+16p+4q+48=0$

$4q=-16p-112$

$q=-4p-28$ A1

$f(-3)=105$

$(-3)^3+p(-3)^2+q(-3)+48=105$

$-27+9p-3q+48=105$

$\therefore -27+9p-3(-4p-28)+48=105$ (M1) for substitution

$-27+9p+12p+84+48=105$

$21p=0$

$p=0$ A1

$q=-4(0)-28$

$q=-28$ A1

[5]

3. (a) $p(5)=0$

$5^3+5m-30=0$ (M1) for factor theorem

$5m=-95$

$m=-19$ A1

[2]

(b) $p(x)=x^3-19x-30$

$$\begin{array}{r} x^2+5x+6 \\ x-5\overline{)\,x^3+0x^2-19x-30} \\ \underline{x^3-5x^2} \qquad\qquad \\ 5x^2-19x \qquad \\ \underline{5x^2-25x} \qquad \\ 6x-30 \\ \underline{6x-30} \end{array}$$

M1

By using the long division,

$p(x)=(x-5)(x^2+5x+6)$ A1

$p(x)=(x-5)(x+2)(x+3)$ A1

[3]

4. (a) $q(-3)=0$

$2(-3)^3+(k+9)(-3)^2+k(-3)+(k+1)=0$ (M1) for factor theorem

$-54+9k+81-3k+k+1=0$

$7k=-28$

$k=-4$ A1

[2]

(b) $q(x)=2x^3+5x^2-4x-3$

$$\begin{array}{r} 2x^2-x-1 \\ x+3\overline{)\,2x^3+5x^2-4x-3} \\ \underline{2x^3+6x^2} \qquad\qquad \\ -x^2-4x \qquad \\ \underline{-x^2-3x} \qquad \\ -x-3 \\ \underline{-x-3} \end{array}$$

M1

By using the long division,

$p(x)=(x+3)(2x^2-x-1)$ A1

$p(x)=(x+3)(2x+1)(x-1)$ A1

[3]

Exercise 8

1. $2x^3+30x^2+kx+20=x-5$

$2x^3+30x^2+(k-1)x+25=0$ (M1) for valid approach

$r_1r_2+r_2r_3+r_1r_3=2r_1r_2r_3$

$\therefore \frac{k-1}{2}=-\frac{25}{2}$ M1A2

$k-1=-25$

$k=-24$ A1

[5]

2. $x^4+kx^3+3x^2+2x+1=-x^4+4x^3+13$

$2x^4+(k-4)x^3+3x^2+2x-12=0$ (M1) for valid approach

$r_1r_2r_3r_4+r_1+r_2+r_3+r_4=0$

$\therefore \frac{-12}{2}+\left(-\frac{k-4}{2}\right)=0$ M1A2

$-12-k+4=0$

$k=-8$ A1

[5]

3. (a) $\frac{1}{r_1r_2}+\frac{1}{r_2r_3}+\frac{1}{r_1r_3}=\frac{r_3}{r_1r_2r_3}+\frac{r_1}{r_1r_2r_3}+\frac{r_2}{r_1r_2r_3}$ (M1) for valid approach

$\frac{1}{r_1r_2}+\frac{1}{r_2r_3}+\frac{1}{r_1r_3}=\frac{r_1+r_2+r_3}{r_1r_2r_3}$

$\frac{1}{r_1r_2}+\frac{1}{r_2r_3}+\frac{1}{r_1r_3}=\frac{-\frac{-18}{6}}{-\frac{72}{6}}$ A2

$\frac{1}{r_1r_2}+\frac{1}{r_2r_3}+\frac{1}{r_1r_3}=-\frac{1}{4}$ A1

[4]

(b) The sum of the roots

$=-\frac{-18}{6}+3(3)$ M1

$=12$ A1

[2]

4. (a) $\frac{1}{r_1}+\frac{1}{r_2}+\frac{1}{r_3}=\frac{r_2r_3}{r_1r_2r_3}+\frac{r_1r_3}{r_1r_2r_3}+\frac{r_1r_2}{r_1r_2r_3}$ (M1) for valid approach

$\frac{1}{r_1}+\frac{1}{r_2}+\frac{1}{r_3}=\frac{r_2r_3+r_1r_3+r_1r_2}{r_1r_2r_3}$

$\frac{1}{r_1}+\frac{1}{r_2}+\frac{1}{r_3}=\frac{\frac{-10}{-1}}{-\frac{-15}{-1}}$ A2

$\frac{1}{r_1}+\frac{1}{r_2}+\frac{1}{r_3}=-\frac{2}{3}$ A1

[4]

(b) The product of the roots

$=-\frac{-15}{-1}$ M1

$=-15$ A1

[2]

Exercise 9

1. (a) The required real root

$= 1.25 - 6$ (M1) for valid approach

$= -4.75$ A1

[2]

(b) The required real root

$= -1(1.25)$ (M1) for valid approach

$= -1.25$ A1

[2]

2. (a) The required real root

$= -1(-2.25)$ (M1) for valid approach

$= 2.25$ A1

[2]

(b) The required real root

$= \frac{-2.25}{3}$ (M1) for valid approach

$= -0.75$ A1

[2]

3. $g(x) = f(x-4)$

$g(x) = -(x-4)^4 + 4(x-4)^3 + 5(x-4) + 3$ (A1) for substitution

$g(x) = -(x^4 - 16x^3 + 96x^2 - 256x + 256)$
$+4(x^3 - 12x^2 + 48x - 64) + 5x - 20 + 3$ M1A1

$g(x) = -x^4 + 16x^3 - 96x^2 + 256x - 256$
$+4x^3 - 48x^2 + 192x - 256 + 5x - 20 + 3$ M1

$g(x) = -x^4 + 20x^3 - 144x^2 + 453x - 529$ A1

[5]

4. $g(x) = 2f(x+1)$

$g(x) = 2(2(x+1)^4 + 5(x+1)^2 - 2)$ (A1) for substitution

$g(x) = 2(2(x^4 + 4x^3 + 6x^2 + 4x + 1) + 5(x^2 + 2x + 1) - 2)$ M1A1

$g(x) = 2(2x^4 + 8x^3 + 12x^2 + 8x + 2 + 5x^2 + 10x + 5 - 2)$ M1

$g(x) = 2(2x^4 + 8x^3 + 17x^2 + 18x + 5)$

$g(x) = 4x^4 + 16x^3 + 34x^2 + 36x + 10$ A1

[5]

Exercise 10

1. $-2x^3 - 3x^2 + 12x + r - 7 = 0$

$r = 2x^3 + 3x^2 - 12x + 7$ (M1) for valid approach

By considering the graph of $y = 2x^3 + 3x^2 - 12x + 7$, the local maximum is $(-2, 27)$ and the local minimum is $(1, 0)$. (M1) for valid approach

Thus, $0 < r < 27$. A2

[4]

2. $x^4 - 8x^3 + 16x^2 = k - 4x$

$x^4 - 8x^3 + 16x^2 + 4x = k$ (M1) for valid approach

By considering the graph of $y = x^4 - 8x^3 + 16x^2 + 4x$, the global minimum is $(-0.114908, -0.236058)$. (M1) for valid approach

Thus, $k < -0.236$. A2

[4]

3. $x^2 + 180 = \dfrac{r + 24x^2}{x}$

$x^3 + 180x = r + 24x^2$

$x^3 - 24x^2 + 180x = r$ (M1) for valid approach

By considering the graph of $y = x^3 - 24x^2 + 180x$, the local maximum is $(6, 432)$ and the local minimum is $(10, 400)$. (M1) for valid approach

Thus, $r \le 400$ or $r \ge 432$. A2

[4]

4. $8x^2(2x+1) = k + 192x + x^4$

$16x^3 + 8x^2 = k + 192x + x^4$

$-x^4 + 16x^3 + 8x^2 - 192x = k$ (M1) for valid approach

By considering the graph of $y = -x^4 + 16x^3 + 8x^2 - 192x$, the local maxima are $(-2, 272)$ and $(12, 5760)$, and the local minimum is $(2, -240)$. (M1) for valid approach

Thus, $-240 < k < 272$. A2

[4]

Exercise 11

1. (a) Let $\frac{x+1}{(x-4)(3x-1)} \equiv \frac{A}{x-4} + \frac{B}{3x-1}$, where A and B are constants.

$$\frac{x+1}{(x-4)(3x-1)} \equiv \frac{A(3x-1)}{(x-4)(3x-1)} + \frac{B(x-4)}{(x-4)(3x-1)}$$ M1

$$\frac{x+1}{(x-4)(3x-1)} \equiv \frac{3Ax-A+Bx-4B}{(x-4)(3x-1)}$$

$x+1 \equiv (3A+B)x+(-A-4B)$ A1

$1=3A+B$

$B=1-3A$

$1=-A-4B$

$\therefore 1=-A-4(1-3A)$ A1

$1=-A-4+12A$

$5=11A$

$A=\frac{5}{11}$

$\therefore B=1-3\left(\frac{5}{11}\right)$

$B=-\frac{4}{11}$

$\therefore \frac{x+1}{(x-4)(3x-1)} \equiv \frac{5}{11(x-4)} - \frac{4}{11(3x-1)}$ A1

[4]

(b) -1 A1

[1]

(c) $x=4$, $x=\frac{1}{3}$ A2

[2]

2. (a) $a=-2$, $b=-\frac{1}{2}$ A2

[2]

(b) Let $\frac{3-x}{2x^2+5x+2}\equiv\frac{A}{2x+1}+\frac{B}{x+2}$, where A and B are constants.

$\frac{3-x}{2x^2+5x+2}\equiv\frac{A(x+2)}{(2x+1)(x+2)}+\frac{B(2x+1)}{(2x+1)(x+2)}$ M1

$\frac{3-x}{2x^2+5x+2}\equiv\frac{Ax+2A+2Bx+B}{(2x+1)(x+2)}$

$-x+3\equiv(A+2B)x+(2A+B)$ A1

$-1=A+2B$

$A=-1-2B$

$3=2A+B$

$\therefore 3=2(-1-2B)+B$ A1

$3=-2-4B+B$

$5=-3B$

$B=-\frac{5}{3}$

$\therefore A=-1-2\left(-\frac{5}{3}\right)$

$A=\frac{7}{3}$

$\therefore \frac{3-x}{2x^2+5x+2}\equiv\frac{7}{3(2x+1)}-\frac{5}{3(x+2)}$ A1

[4]

3. (a) Let $\frac{x^2-x-1}{(x+3)(x+7)} \equiv A+\frac{B}{x+3}+\frac{C}{x+7}$, where A, B and C are constants.

$$\frac{x^2-x-1}{(x+3)(x+7)} \equiv \frac{A(x+3)(x+7)}{(x+3)(x+7)} + \frac{B(x+7)}{(x+3)(x+7)} + \frac{C(x+3)}{(x+3)(x+7)}$$ M1

$$\frac{x^2-x-1}{(x+3)(x+7)} \equiv \frac{Ax^2+10Ax+21A+Bx+7B+Cx+3C}{(x+3)(x+7)}$$

$x^2-x-1 \equiv Ax^2+(10A+B+C)x+(21A+7B+3C)$ A1

$A=1$ A1

$-1=10(1)+B+C$

$C=-11-B$

$-1=21A+7B+3C$

$\therefore -1=21(1)+7B+3(-11-B)$ A1

$-1=21+7B-33-3B$

$11=4B$

$B=\frac{11}{4}$ A1

$\therefore C=-11-\frac{11}{4}$

$C=-\frac{55}{4}$ A1

[6]

(b) $y=1$ A1

[1]

4. (a) Let $\frac{x^2+3}{(2-x)(5-3x)} \equiv A+\frac{B}{2-x}+\frac{C}{5-3x}$, where A, B and C are constants.

$$\frac{x^2+3}{(2-x)(5-3x)} \equiv \frac{A(2-x)(5-3x)}{(2-x)(5-3x)}$$ M1

$$+\frac{B(5-3x)}{(2-x)(5-3x)}+\frac{C(2-x)}{(2-x)(5-3x)}$$

$$\frac{x^2+3}{(2-x)(5-3x)}$$

$$\equiv \frac{10A-11Ax+3Ax^2+5B-3Bx+2C-Cx}{(2-x)(5-3x)}$$

$x^2+3 \equiv 3Ax^2+(-11A-3B-C)x+(10A+5B+2C)$ A1

$3A=1$

$A=\frac{1}{3}$ A1

$0=-11\left(\frac{1}{3}\right)-3B-C$

$C=-\frac{11}{3}-3B$

$3=10A+5B+2C$

$\therefore 3=10\left(\frac{1}{3}\right)+5B+2\left(-\frac{11}{3}-3B\right)$ A1

$3=\frac{10}{3}+5B-\frac{22}{3}-6B$

$3=-4-B$

$B=-7$ A1

$\therefore C=-\frac{11}{3}-3(-7)$

$C=\frac{52}{3}$ A1

[6]

(b) $g(x)=\frac{(2-x)(5-3x)}{x^2+3}$

The discriminant of x^2+3

$=0^2-4(1)(3)$ A1

$=-12$

<0

Therefore, the denominator is always nonzero.

Thus, $g(x)$ has no vertical asymptote. AG

[1]

Chapter 3 Solution

Exercise 12

1. (a) $a+b+c=998$ A1

$9a+3b+c=982$ A1

$36a+6b+c=928$ A1

[3]

(b) $a=-2$, $b=0$ and $c=1000$

For any one correct answer A1

For all correct answers A1

[2]

2. (a) $x+y+z=8400$ A1

$x+z=y-6288$ A1

[2]

(b) $42x+84y+21z=655872$ A1

By solving the system $\begin{cases} x+y+z=8400 \\ x-y+z=-6288 \\ 42x+84y+21z=655872 \end{cases}$,

$x=800$, $y=7344$ and $z=256$.

For any one correct answer A1

For all correct answers A1

[3]

3. (a) $\begin{cases} 10a+12b+13c=150 \\ 14a+8b+19c=178 \\ 22a+23b+7c=230 \end{cases}$ M1A1

By solving the system $\begin{cases} 10a+12b+13c=150 \\ 14a+8b+19c=178 \\ 22a+23b+7c=230 \end{cases}$,

$a=5$, $b=4$ and $c=4$.

For any one correct answer A1

For all correct answers A1

[4]

(b) The total price

$=5(30)+4(0)+4(35)$ M1

$=\$290$ A1

[2]

4. (a) $\begin{cases} 30x+16y=152 \\ 23x+15y+8z=114 \\ 11x+17y+18z=60 \end{cases}$ M1A1

By solving the system $\begin{cases} 30x+16y=152 \\ 23x+15y+8z=114 \\ 11x+17y+18z=60 \end{cases}$,

$x=4$, $y=2$ and $z=-1$.

For any one correct answer A1

For all correct answers A1

[4]

(b) A team drops 1 point for losing a game. A1

[1]

Exercise 13

1. (a) $\begin{cases} x+3y-2z=3 \\ 2x+y-z=1 \\ -x+2y+az=2 \end{cases}$

$\rightarrow \begin{cases} x+3y-2z=3 \\ -5y+3z=-5 \quad (R_2-2R_1 \ \& \ R_3+R_1) \\ 5y+(a-2)z=5 \end{cases}$ M1A1

$\rightarrow \begin{cases} x+3y-2z=3 \\ -5y+3z=-5 \ (R_3+R_2) \\ (a+1)z=0 \end{cases}$ A1

The system has an infinite number of solutions.

$\therefore 0=a+1$

$a=-1$ A1

[4]

(b) The system has a unique solution when $a \neq -1$. R1

$\therefore z=0$

$-5y+3(0)=-5$

$y=1$

$x+3(1)-2(0)=3$

$x=0$

Thus, $x=0$, $y=1$ and $z=0$. A2

[3]

2. (a) $\begin{cases} x+y+z=3 \\ 2x+y-3z=-2 \\ x-2y+az=-21 \end{cases}$

$\rightarrow \begin{cases} x+y+z=3 \\ -y-5z=-8 \\ -3y+(a-1)z=-24 \end{cases}$ $(R_2-2R_1 \,\&\, R_3-R_1)$ M1A1

$\rightarrow \begin{cases} x+y+z=3 \\ -y-5z=-8 (R_3-3R_2) \\ (a+14)z=0 \end{cases}$ A1

The system has an infinite number of solutions.

$\therefore 0=a+14$

$a=-14$ A1

[4]

(b) The system has a unique solution when $a=6$. R1

$\therefore z=0$

$-y-5(0)=-8$

$y=8$

$x+8+0=3$

$x=-5$

Thus, $x=-5$, $y=8$ and $z=0$. A2

[3]

3. (a) (i) $\begin{cases} 2x - y + z = 1 \\ -x + y + az = 0 \\ x - 2y - 2z = b \end{cases}$

$\to \begin{cases} 2x - y + z = 1 \\ 0.5y + (a + 0.5)z = 0.5 \\ -1.5y - 2.5z = b - 0.5 \end{cases}$ M1A1

$(R_2 + 0.5R_1 \ \& \ R_3 - 0.5R_1)$

$\to \begin{cases} 2x - y + z = 1 \\ 0.5y + (a + 0.5)z = 0.5 \ (R_3 + 3R_2) \\ (3a - 1)z = b + 1 \end{cases}$ A1

The system has no solutions when $3a - 1 = 0$ and $b + 1 \neq 0$.

$a = \frac{1}{3}$ and $b \neq -1$ A1

(ii) The system has a unique solution when $3a - 1 \neq 0$.

$a \neq \frac{1}{3}$ A1

(iii) The system has an infinite number of solutions when $3a - 1 = 0$ and $b + 1 = 0$.

$a = \frac{1}{3}$ and $b = -1$ A1

[6]

(b) $(3(0) - 1)z = 0 + 1$

$z = -1$

$0.5y + (0 + 0.5)(-1) = 0.5$

$y = 2$

$2x - 2 + (-1) = 1$

$x = 2$

Thus, $x = 2$, $y = 2$ and $z = -1$. A2

[2]

4. (a) (i) $\begin{cases} 8x+3y+z=2 \\ 4x-4y+6z=1 \\ 4x+y+az=b \end{cases}$

$\rightarrow \begin{cases} 8x+3y+z=2 \\ -5.5y+5.5z=0 \\ 5y+(a-6)z=b-1 \end{cases}$ M1A1

$(R_2-0.5R_1 \ \& \ R_3-R_2)$

$\rightarrow \begin{cases} 8x+3y+z=2 \\ -5.5y+5.5z=0 \, (R_3+\frac{10}{11}R_2) \\ (a-1)z=b-1 \end{cases}$ A1

The system has no solutions when
$a-1=0$ and $b-1\neq 0$.
$a=1$ and $b\neq 1$ A1

(ii) The system has a unique solution when
$a-1\neq 0$.
$a\neq 1$ A1

(iii) The system has an infinite number of solutions when $a-1=0$ and $b-1=0$.
$a=1$ and $b=1$ A1

[6]

(b) The system has an infinite number of solutions.
$-5.5y+5.5t=0$
$-y+t=0$
$y=t$ A1
$8x+3t+t=2$
$x=0.25-0.5t$
Thus, $x=0.25-0.5t$, $y=t$ and $z=t$, where $t\in\mathbb{R}$. A1

[2]

Chapter 4 Solution

Exercise 14

1. (a) $\sqrt[3]{1+2x}=(1+2x)^{\frac{1}{3}}$

$\sqrt[3]{1+2x}=1+\left(\frac{1}{3}\right)(2x)+\frac{\left(\frac{1}{3}\right)\left(-\frac{2}{3}\right)}{2!}(2x)^2$

$+\frac{\left(\frac{1}{3}\right)\left(-\frac{2}{3}\right)\left(-\frac{5}{3}\right)}{3!}(2x)^3+\cdots$ M1A1

$\sqrt[3]{1+2x}=1+\frac{2}{3}x-\frac{4}{9}x^2+\frac{40}{81}x^3+\cdots$ A1

[3]

(b) $\sqrt[3]{1.06}=\sqrt[3]{1+2(0.03)}$

$\sqrt[3]{1.06}\approx 1+\frac{2}{3}(0.03)-\frac{4}{9}(0.03)^2+\frac{40}{81}(0.03)^3$ M1

$\sqrt[3]{1.06}\approx 1.019613333$

$\sqrt[3]{1.06}\approx 1.02$ A1

[2]

2. (a)

$$(1+bx)^{-\frac{2}{3}} = 1+\left(-\frac{2}{3}\right)(bx)+\frac{\left(-\frac{2}{3}\right)\left(-\frac{5}{3}\right)}{2!}(bx)^2$$ M1A1

$$+\frac{\left(-\frac{2}{3}\right)\left(-\frac{5}{3}\right)\left(-\frac{8}{3}\right)}{3!}(bx)^3+\cdots$$

$$(1+bx)^{-\frac{2}{3}} = 1-\frac{2}{3}bx+\frac{5}{9}b^2x^2-\frac{40}{81}b^3x^3+\cdots$$

$$\therefore -\frac{40}{81}b^3 = -\frac{2560}{81}$$ (A1) for correct equation

$$b^3 = 64$$

$$b = 4$$ A1

[4]

(b) The coefficient of x^2

$$=\frac{5}{9}(4)^2$$ A1

$$=\frac{80}{9}$$ A1

[2]

3. (a) $\sqrt{4-x}=\sqrt{4\left(1-\frac{1}{4}x\right)}$

$\sqrt{4-x}=2\left(1+\left(-\frac{1}{4}x\right)\right)^{\frac{1}{2}}$

$$\sqrt{4-x}=2\left(\begin{array}{l}1+\left(\frac{1}{2}\right)\left(-\frac{1}{4}x\right)+\frac{\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)}{2!}\left(-\frac{1}{4}x\right)^2\\+\frac{\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)}{3!}\left(-\frac{1}{4}x\right)^3+\cdots\end{array}\right)$$ M1A1

$\sqrt{4-x}=2\left(1-\frac{1}{8}x-\frac{1}{128}x^2-\frac{1}{1024}x^3+\cdots\right)$

$\sqrt{4-x}=2-\frac{1}{4}x-\frac{1}{64}x^2-\frac{1}{512}x^3+\cdots$ A1

[3]

(b) $\sqrt{\frac{15}{4}}=\sqrt{4-\frac{1}{4}}$

$\sqrt{\frac{15}{4}}\approx 2-\frac{1}{4}\left(\frac{1}{4}\right)-\frac{1}{64}\left(\frac{1}{4}\right)^2-\frac{1}{512}\left(\frac{1}{4}\right)^3$ M1A1

$\sqrt{\frac{15}{4}}\approx 1.93649292$

$\therefore \sqrt{15}\approx(1.93649292)(\sqrt{4})$ A1

$\sqrt{15}\approx 3.87298584$

$\sqrt{15}\approx 3.87$ A1

[4]

4. (a) $\sqrt[4]{81+ax} = \sqrt[4]{81\left(1+\frac{a}{81}x\right)}$

$\sqrt[4]{81+ax} = 3\left(1+\left(\frac{a}{81}x\right)\right)^{\frac{1}{4}}$

$\sqrt[4]{81+ax} = 3\left(1+\left(\frac{1}{4}\right)\left(\frac{a}{81}x\right)+\frac{\left(\frac{1}{4}\right)\left(-\frac{3}{4}\right)}{2!}\left(\frac{a}{81}x\right)^2 + \cdots\right)$ M1A1

$\sqrt[4]{81+ax} = 3\left(1+\frac{a}{324}x-\frac{a^2}{69984}x^2+\cdots\right)$

$\sqrt[4]{81+ax} = 3+\frac{a}{108}x-\frac{a^2}{23328}x^2+\cdots$ A1

[3]

(b) $\left(\frac{a}{108}\right)\left(-\frac{a^2}{23328}\right) = -0.5$ (A1) for correct equation

$-\frac{a^3}{2519424} = -0.5$

$a^3 = 1259712$

$a = 108$ A1

[2]

(c) $-1 < \frac{108}{81}x < 1$

$-\frac{3}{4} < x < \frac{3}{4}$ A2

[2]

Exercise 15

1. (a) $\frac{1}{(1+3x)^3}=(1+3x)^{-3}$

$\frac{1}{(1+3x)^3}=1+(-3)(3x)+\frac{(-3)(-4)}{2!}(3x)^2$

$+\frac{(-3)(-4)(-5)}{3!}(3x)^3+\cdots$ M1A1

$\frac{1}{(1+3x)^3}=1-9x+54x^2-270x^3+\cdots$ A1

[3]

(b) $0.97^{-3}=\frac{1}{(1+3(-0.01))^3}$

$0.97^{-3}\approx 1-9(-0.01)+54(-0.01)^2-270(-0.01)^3$ M1

$0.97^{-3}\approx 1.09567$

$0.97^{-3}\approx 1.10$ A1

[2]

2. (a) $(1+bx)^{-2}=1+(-2)(bx)+\frac{(-2)(-3)}{2!}(bx)^2$

$+\frac{(-2)(-3)(-4)}{3!}(bx)^3+\cdots$ M1A1

$(1+bx)^{-2}=1-2bx+3b^2x^2-4b^3x^3+\cdots$

$\therefore -4b^3=2048$ (A1) for correct equation

$b^3=-512$

$b=-8$ A1

[4]

(b) The coefficient of x^2

$=3(-8)^2$ A1

$=192$ A1

[2]

3. (a)

$$\frac{1}{4+12x+9x^2}=\frac{1}{(2+3x)^2}$$

$$\frac{1}{4+12x+9x^2}=\frac{1}{2^2\left(1+\frac{3}{2}x\right)^2}$$

$$\frac{1}{4+12x+9x^2}=\frac{1}{4}\left(1+\frac{3}{2}x\right)^{-2}$$

$$\frac{1}{4+12x+9x^2}=\frac{1}{4}\left(\begin{array}{l}1+(-2)\left(\frac{3}{2}x\right)+\frac{(-2)(-3)}{2!}\left(\frac{3}{2}x\right)^2\\+\frac{(-2)(-3)(-4)}{3!}\left(\frac{3}{2}x\right)^3+\cdots\end{array}\right)$$ M1A1

$$\frac{1}{4+12x+9x^2}=\frac{1}{4}\left(1-3x+\frac{27}{4}x^2-\frac{27}{2}x^3+\cdots\right)$$

$$\frac{1}{4+12x+9x^2}=\frac{1}{4}-\frac{3}{4}x+\frac{27}{16}x^2-\frac{27}{8}x^3+\cdots$$ A1

[3]

(b)

$$\frac{1}{2.15^2}=\frac{1}{(2+3(0.05))^2}$$

$$\frac{1}{2.15^2}\approx\frac{1}{4}-\frac{3}{4}(0.05)+\frac{27}{16}(0.05)^2-\frac{27}{8}(0.05)^3$$ M1A1

$$\frac{1}{2.15^2}\approx 0.216296875$$

$$\therefore -\frac{7}{2.15^2}\approx(0.216296875)(-7)$$ A1

$$-\frac{7}{2.15^2}\approx -1.514078125$$

$$-\frac{7}{2.15^2}\approx -1.51$$ A1

[4]

4. (a) $\frac{2-x}{a+x}=(2-x)(a+x)^{-1}$

$$\frac{2-x}{a+x}=(2-x)(a)^{-1}\left(1+\frac{1}{a}x\right)^{-1}$$

$$\frac{2-x}{a+x}=\frac{2-x}{a}\left(1+(-1)\left(\frac{1}{a}x\right)+\frac{(-1)(-2)}{2!}\left(\frac{1}{a}x\right)^{2}+\cdots\right)$$ M1A1

$$\frac{2-x}{a+x}=\left(\frac{2}{a}-\frac{x}{a}\right)\left(1-\frac{1}{a}x+\frac{1}{a^{2}}x^{2}+\cdots\right)$$

$$\frac{2-x}{a+x}=\frac{2}{a}-\frac{2}{a^{2}}x+\frac{2}{a^{3}}x^{2}-\frac{1}{a}x+\frac{1}{a^{2}}x^{2}-\frac{1}{a^{3}}x^{3}+\cdots$$

$$\frac{2-x}{a+x}=\frac{2}{a}+\left(-\frac{1}{a}-\frac{2}{a^{2}}\right)x+\left(\frac{1}{a^{2}}+\frac{2}{a^{3}}\right)x^{2}+\cdots$$ A1

[3]

(b) (i) $\frac{2}{a}+\left(-\frac{1}{a}-\frac{2}{a^{2}}\right)=\frac{1}{9}$ (A1) for correct equation

$$\frac{1}{a}-\frac{2}{a^{2}}=\frac{1}{9}$$

$$\frac{a-2}{a^{2}}=\frac{1}{9}$$

$$9a-18=a^{2}$$

$$a^{2}-9a+18=0$$

$$(a-3)(a-6)=0$$

$a=3$ or $a=6$ (*Rejected*) A1

(ii) $\frac{5}{27}$ A1

[3]

(c) $-1<\frac{1}{3}x<1$

$-3<x<3$ A2

[2]

Exercise 16

1. (a) Let $\frac{2}{(2+x)(5+x)} \equiv \frac{A}{2+x} + \frac{B}{5+x}$, where A and B are constants.

$$\frac{2}{(2+x)(5+x)} \equiv \frac{A(5+x)}{(2+x)(5+x)} + \frac{B(2+x)}{(2+x)(5+x)}$$ M1

$$\frac{2}{(2+x)(5+x)} \equiv \frac{5A+Ax+2B+Bx}{(2+x)(5+x)}$$

$$2 \equiv (A+B)x + (5A+2B)$$ A1

$$0 = A+B$$

$$-B = A$$

$$2 = 5A+2B$$

$$\therefore 2 = 5(-B)+2B$$ A1

$$2 = -5B+2B$$

$$2 = -3B$$

$$B = -\frac{2}{3}$$

$$\therefore A = \frac{2}{3}$$

$$\therefore \frac{2}{(2+x)(5+x)} \equiv \frac{2}{3(2+x)} - \frac{2}{3(5+x)}$$ A1

[4]

(b) $$\frac{2}{(2+x)(5+x)}=\frac{2}{3(2+x)}-\frac{2}{3(5+x)}$$

$$\frac{2}{(2+x)(5+x)}=\frac{2}{6}\left(1+\frac{1}{2}x\right)^{-1}-\frac{2}{15}\left(1+\frac{1}{5}x\right)^{-1}$$

$$\frac{2}{(2+x)(5+x)}$$

$$=\frac{1}{3}\left(1+(-1)\left(\frac{1}{2}x\right)+\frac{(-1)(-2)}{2!}\left(\frac{1}{2}x\right)^2+\cdots\right)$$ M1A1

$$-\frac{2}{15}\left(1+(-1)\left(\frac{1}{5}x\right)+\frac{(-1)(-2)}{2!}\left(\frac{1}{5}x\right)^2+\cdots\right)$$

$$\frac{2}{(2+x)(5+x)}=\frac{1}{3}\left(1-\frac{1}{2}x+\frac{1}{4}x^2+\cdots\right)$$

$$-\frac{2}{15}\left(1-\frac{1}{5}x+\frac{1}{25}x^2+\cdots\right)$$

$$\frac{2}{(2+x)(5+x)}$$ A1

$$=\frac{1}{3}-\frac{1}{6}x+\frac{1}{12}x^2-\frac{2}{15}+\frac{2}{75}x-\frac{2}{375}x^2+\cdots$$

$$\frac{2}{(2+x)(5+x)}=\frac{1}{5}-\frac{7}{50}x+\frac{39}{500}x^2+\cdots$$

The required sum

$$=-\frac{7}{50}+\frac{39}{500}$$

$$=-\frac{31}{500}$$ A1

[4]

2. (a) Let $\frac{1+x}{(4+x)^2} \equiv \frac{A}{4+x} + \frac{B}{(4+x)^2}$, where A and B are constants.

$\frac{1+x}{(4+x)^2} \equiv \frac{A(4+x)}{(4+x)^2} + \frac{B}{(4+x)^2}$ M1

$\frac{1+x}{(4+x)^2} \equiv \frac{4A+Ax+B}{(4+x)^2}$

$1+x \equiv Ax + (4A+B)$ A1

$A = 1$

$1 = 4A + B$

$\therefore 1 = 4(1) + B$ A1

$1 = 4 + B$

$B = -3$

$\therefore \frac{1+x}{(4+x)^2} \equiv \frac{1}{4+x} - \frac{3}{(4+x)^2}$ A1

[4]

(b) $\dfrac{1+x}{(4+x)^2} = \dfrac{1}{4+x} - \dfrac{3}{(4+x)^2}$

$$\frac{1+x}{(4+x)^2} = \frac{1}{4}\left(1+\frac{1}{4}x\right)^{-1} - \frac{3}{16}\left(1+\frac{1}{4}x\right)^{-2}$$

$$\frac{1+x}{(4+x)^2} = \frac{1}{4}\left(\begin{array}{l} 1+(-1)\left(\frac{1}{4}x\right)+\frac{(-1)(-2)}{2!}\left(\frac{1}{4}x\right)^2 \\ +\frac{(-1)(-2)(-3)}{3!}\left(\frac{1}{4}x\right)^3+\cdots \end{array}\right)$$

M1A1

$$-\frac{3}{16}\left(\begin{array}{l} 1+(-2)\left(\frac{1}{4}x\right)+\frac{(-2)(-3)}{2!}\left(\frac{1}{4}x\right)^2 \\ +\frac{(-2)(-3)(-4)}{3!}\left(\frac{1}{4}x\right)^3+\cdots \end{array}\right)$$

$$\frac{1+x}{(4+x)^2} = \frac{1}{4}\left(1-\frac{1}{4}x+\frac{1}{16}x^2-\frac{1}{64}x^3+\cdots\right)$$

M1

$$-\frac{3}{16}\left(1-\frac{1}{2}x+\frac{3}{16}x^2-\frac{1}{16}x^3+\cdots\right)$$

$$\frac{1+x}{(4+x)^2} = \frac{1}{4}-\frac{1}{16}x+\frac{1}{64}x^2-\frac{1}{256}x^3$$

A1

$$-\frac{3}{16}+\frac{3}{32}x-\frac{9}{256}x^2+\frac{3}{256}x^3+\cdots$$

$$\frac{1+x}{(4+x)^2} = \frac{1}{16}+\frac{1}{32}x-\frac{5}{256}x^2+\frac{1}{128}x^3+\cdots$$

Thus, the coefficient of x^3 is $\dfrac{1}{128}$. A1

[5]

3. (a) Let $\frac{9x}{(3+x)(6-x)} \equiv \frac{A}{3+x} + \frac{B}{6-x}$, where A and B are constants.

$\frac{9x}{(3+x)(6-x)} \equiv \frac{A(6-x)}{(3+x)(6-x)} + \frac{B(3+x)}{(3+x)(6-x)}$ M1

$\frac{9x}{(3+x)(6-x)} \equiv \frac{6A - Ax + 3B + Bx}{(3+x)(6-x)}$

$9x \equiv (-A+B)x + (6A+3B)$ A1

$9 = -A + B$

$B = 9 + A$

$0 = 6A + 3B$

$\therefore 0 = 6A + 3(9+A)$ A1

$0 = 9A + 27$

$-27 = 9A$

$A = -3 \therefore B = 6$

$\therefore \frac{9x}{(3+x)(6-x)} \equiv -\frac{3}{3+x} + \frac{6}{6-x}$ A1

$\frac{9x}{(3+x)(6-x)} = -\frac{3}{3}\left(1+\frac{1}{3}x\right)^{-1} + \frac{6}{6}\left(1-\frac{1}{6}x\right)^{-1}$

$\frac{9x}{(3+x)(6-x)}$

$= -\left(1 + (-1)\left(\frac{1}{3}x\right) + \frac{(-1)(-2)}{2!}\left(\frac{1}{3}x\right)^2 + \cdots\right)$ M1A1

$+\left(1 + (-1)\left(-\frac{1}{6}x\right) + \frac{(-1)(-2)}{2!}\left(-\frac{1}{6}x\right)^2 + \cdots\right)$

$\frac{9x}{(3+x)(6-x)} = -\left(1 - \frac{1}{3}x + \frac{1}{9}x^2 + \cdots\right)$

$+\left(1 + \frac{1}{6}x + \frac{1}{36}x^2 + \cdots\right)$

$\frac{9x}{(3+x)(6-x)} = -1 + \frac{1}{3}x - \frac{1}{9}x^2 + 1 + \frac{1}{6}x + \frac{1}{36}x^2 + \cdots$

$\frac{9x}{(3+x)(6-x)} = \frac{1}{2}x - \frac{1}{12}x^2 + \cdots$ A1

[7]

(b) $-1 < \frac{1}{3}x < 1$ and $-1 < -\frac{1}{6}x < 1$

$-3 < x < 3$ and $-6 < x < 6$

$\therefore -3 < x < 3$ A2

[2]

4. Let $\dfrac{1+5x}{(3+kx)^2} \equiv \dfrac{A}{3+kx} + \dfrac{B}{(3+kx)^2}$, where A and B are constants.

$$\frac{1+5x}{(3+kx)^2} \equiv \frac{A(3+kx)}{(3+kx)^2} + \frac{B}{(3+kx)^2}$$ M1

$$\frac{1+5x}{(3+kx)^2} \equiv \frac{3A+Akx+B}{(3+kx)^2}$$

$$1+5x \equiv Akx+(3A+B)$$ A1

$$5 = Ak$$

$$A = \frac{5}{k}$$

$$1 = 3A+B$$

$$\therefore 1 = 3\left(\frac{5}{k}\right)+B$$ A1

$$1 = \frac{15}{k}+B$$

$$B = \frac{k-15}{k}$$

$$\therefore \frac{1+5x}{(3+kx)^2} \equiv \frac{5}{k(3+kx)} + \frac{k-15}{k(3+kx)^2}$$

$$\frac{1+5x}{(3+kx)^2} = \frac{5}{3k}\left(1+\frac{k}{3}x\right)^{-1} + \frac{k-15}{3k}\left(1+\frac{k}{3}x\right)^{-2}$$

$$\frac{1+5x}{(3+kx)^2}$$

$$= \frac{5}{3k}\left(1+(-1)\left(\frac{k}{3}x\right)+\frac{(-1)(-2)}{2!}\left(\frac{k}{3}x\right)^2+\cdots\right)$$ M1A1

$$+\frac{k-15}{3k}\left(1+(-2)\left(\frac{k}{3}x\right)+\frac{(-2)(-3)}{2!}\left(\frac{k}{3}x\right)^2+\cdots\right)$$

$$\frac{1+5x}{(3+kx)^2} = \frac{5}{3k}\left(1-\frac{k}{3}x+\frac{k^2}{9}x^2+\cdots\right)$$

$$+\frac{k-15}{3k}\left(1-\frac{2k}{3}x+\frac{k^2}{3}x^2+\cdots\right)$$

$$\therefore \frac{5}{3k}\left(\frac{k^2}{9}\right)+\frac{k-15}{3k}\left(\frac{k^2}{3}\right) = -\frac{37}{27}$$ A1

$$\frac{5k}{27}+\frac{k^2-15k}{9} = -\frac{37}{27}$$

$$5k+3k^2-45k = -37$$

$$3k^2-40k+37 = 0$$ A1

$(3k-37)(k-1)=0$

$k=\frac{37}{3}$ (*Rejected*) or $k=1$ A1

[8]

Chapter 5 Solution

Exercise 17

1. When $n=1$,

$$\text{L.H.S.}=\sum_{r=1}^{1}(2r+1)^2$$

$$\text{L.H.S.}=9$$

$$\text{R.H.S.}=\frac{(1)(4(1)^2+12(1)+11)}{3}$$

$$\text{R.H.S.}=9$$

Thus, the statement is true when $n=1$. R1

Assume that the statement is true when $n=k$. M1

$$\sum_{r=1}^{k}(2r+1)^2=\frac{k(4k^2+12k+11)}{3}$$

When $n=k+1$,

$$\sum_{r=1}^{k+1}(2r+1)^2=\sum_{r=1}^{k}(2r+1)^2+(2(k+1)+1)^2 \quad \text{M1}$$

$$\sum_{r=1}^{k+1}(2r+1)^2=\frac{k(4k^2+12k+11)}{3}+(4k^2+12k+9) \quad \text{A1}$$

$$\sum_{r=1}^{k+1}(2r+1)^2=\frac{4k^3+12k^2+11k+12k^2+36k+27}{3} \quad \text{A1}$$

$$\sum_{r=1}^{k+1}(2r+1)^2=\frac{4k^3+24k^2+47k+27}{3}$$

$$\sum_{r=1}^{k+1}(2r+1)^2=\frac{(k+1)(4k^2+20k+27)}{3} \quad \text{A1}$$

$$\sum_{r=1}^{k+1}(2r+1)^2=\frac{(k+1)(4(k+1)^2+12(k+1)+11)}{3} \quad \text{A1}$$

Thus, the statement is true when $n=k+1$.

Therefore, the statement is true for all $n\in\mathbb{Z}^+$. R1

[8]

2. When $n=1$,

L.H.S. $=1\times 1!$

L.H.S. $=1$

R.H.S. $=(1+1)!-1$

R.H.S. $=1$

Thus, the statement is true when $n=1$. R1

Assume that the statement is true when $n=k$. M1

$1\times 1!+2\times 2!+\ldots+k\times k!=(k+1)!-1$

When $n=k+1$,

$1\times 1!+2\times 2!+\ldots+k\times k!+(k+1)\times(k+1)!$

$=(k+1)!-1+(k+1)\times(k+1)!$ M1A1

$=(k+1)!(1+(k+1))-1$ A1

$=(k+1)!(k+2)-1$

$=(k+2)!-1$ A1

Thus, the statement is true when $n=k+1$.

Therefore, the statement is true for all $n\in\mathbb{Z}^+$. R1

[7]

3. When $n=1$,

$\text{L.H.S.}=\binom{1}{1}$

$\text{L.H.S.}=1$

$\text{R.H.S.}=\dfrac{1(1+1)}{2}$

$\text{R.H.S.}=1$

Thus, the statement is true when $n=1$. R1

Assume that the statement is true when $n=k$. M1

$$\binom{1}{1}+\binom{2}{1}+\ldots+\binom{k}{1}=\frac{k(k+1)}{2}$$

When $n=k+1$,

$$\binom{1}{1}+\binom{2}{1}+\ldots+\binom{k}{1}+\binom{k+1}{1}$$

$$=\frac{k(k+1)}{2}+\binom{k+1}{1}$$ M1A1

$$=\frac{k(k+1)}{2}+(k+1)$$ A1

$$=(k+1)\left[\frac{k}{2}+1\right]$$

$$=\frac{(k+1)(k+2)}{2}$$

$$=\frac{(k+1)((k+1)+1)}{2}$$ A1

Thus, the statement is true when $n=k+1$.

Therefore, the statement is true for all $n\in\mathbb{Z}^+$. R1

[7]

4. When $n = 1$,

L.H.S. $= 1^3$

L.H.S. $= 1$

R.H.S. $= 1^4$

R.H.S. $= 1$

Thus, the statement is true when $n = 1$. R1

Assume that the statement is true when $n = k$. M1

$1^3 + 2^3 + \ldots + k^3 \le k^4$

When $n = k + 1$,

$1^3 + 2^3 + \ldots + k^3 + (k+1)^3$

$\le k^4 + (k+1)^3$ M1A1

$= k^4 + k^3 + 3k^2 + 3k + 1$ A1

$\le k^4 + 4k^3 + 6k^2 + 4k + 1$ A1

$= (k+1)^4$

Thus, the statement is true when $n = k + 1$.

Therefore, the statement is true for all $n \in \mathbb{Z}^+$. R1

[7]

Exercise 18

1. When $n=1$,

$9^{1+1}+11=92$

$9^{1+1}+11=4(23)$ A1

Thus, the statement is true when $n=1$.

Assume that the statement is true when $n=k$. M1

$9^{k+1}+11=4M$, where $M\in\mathbb{Z}$.

When $n=k+1$,

$9^{(k+1)+1}+11=9(9^{k+1})+11$ M1

$9^{(k+1)+1}+11=9(4M-11)+11$ A1

$9^{(k+1)+1}+11=36M-88$ M1

$9^{(k+1)+1}+11=4(9M-22)$, where $9M-22\in\mathbb{Z}$. A1

Thus, the statement is true when $n=k+1$.

Therefore, the statement is true for all $n\in\mathbb{Z}^+$. R1

[7]

2. When $n=1$,

$9^1-4^1=5$

$9^1-4^1=5(1)$ A1

Thus, the statement is true when $n=1$.

Assume that the statement is true when $n=k$. M1

$9^k-4^k=5M$, where $M\in\mathbb{Z}$.

When $n=k+1$,

$9^{k+1}-4^{k+1}=9(9^k)-4^{k+1}$ M1

$9^{k+1}-4^{k+1}=9(5M+4^k)-4^{k+1}$ A1

$9^{k+1}-4^{k+1}=45M+9(4^k)-4(4^k)$ M1

$9^{k+1}-4^{k+1}=5(9M+4^k)$, where $9M+4^k\in\mathbb{Z}$. A1

Thus, the statement is true when $n=k+1$.

Therefore, the statement is true for all $n\in\mathbb{Z}^+$. R1

[7]

3. When $n=1$,

$1^3-4(1)=-3$

$1^3-4(1)=3(-1)$ A1

Thus, the statement is true when $n=1$.

Assume that the statement is true when $n=k$. M1

$k^3-4k=3M$, where $M\in\mathbb{Z}$.

When $n=k+1$,

$(k+1)^3-4(k+1)=k^3+3k^2+3k+1-4k-4$ M1

$(k+1)^3-4(k+1)=(3M+4k)+3k^2+3k+1-4k-4$ A1

$(k+1)^3-4(k+1)=3M+3k^2+3k-3$ M1

$(k+1)^3-4(k+1)=3(M+k^2+k-1)$, where

$M+k^2+k-1\in\mathbb{Z}$. A1

Thus, the statement is true when $n=k+1$.

Therefore, the statement is true for all $n\in\mathbb{Z}^+$. R1

[7]

4. When $n=1$,

$16^1+12(1)+8=36$

$16^1+12(1)+8=9(4)$ A1

Thus, the statement is true when $n=1$.

Assume that the statement is true when $n=k$. M1

$16^k+12k+8=9M$, where $M\in\mathbb{Z}$.

When $n=k+1$,

$16^{k+1}+12(k+1)+8=16(16^k)+12k+20$ M1

$16^{k+1}+12(k+1)+8=16(9M-12k-8)+12k+20$ A1

$16^{k+1}+12(k+1)+8=144M-180k-108$ M1

$16^{k+1}+12(k+1)+8=9(16M-20k-12)$, where

$16M-20k-12\in\mathbb{Z}$. A1

Thus, the statement is true when $n=k+1$.

Therefore, the statement is true for all $n\in\mathbb{Z}^+$. R1

[7]

Chapter 6 Solution

Exercise 19

1. (a) Suppose N is an odd number. M1

N^2 is an odd number and $8N$ is an even number A1

N^2+8N is an odd number

$N^2+8N+10$ is an odd number, which contradicts with the statement $N^2+8N+10$ is an even number.

Thus, if $N^2+8N+10$ is an even number, then N is also an even number. AG

[2]

(b) Let $P=1$ and $Q=-1$. A1

$|P-Q|=|1-(-1)|$

$|P-Q|=|2|$

$|P-Q|=2$

$|P|-|Q|=|1|-|-1|$ M1

$|P|-|Q|=1-1$

$|P|-|Q|=0$

$\therefore |P-Q|>|P|-|Q|$

Thus, $|P-Q|\le|P|-|Q|$ is not always true. AG

[2]

2. (a) Suppose $\frac{1}{x+2}+\frac{x+4}{2}=0$ has real solution. M1

$2(x+2)\left(\frac{1}{x+2}+\frac{x+4}{2}\right)=0$

$2+(x+2)(x+4)=0$ A1

$2+x^2+6x+8=0$

$x^2+6x+10=0$

$\Delta=6^2-4(1)(10)$ A1

$\Delta=-4$

$\Delta<0$, which contradicts with the statement

$\frac{1}{x+2}+\frac{x+4}{2}=0$ has real solution.

Thus, the equation $\frac{1}{x+2}+\frac{x+4}{2}=0$ has no real

solution for all real values of x. AG

[3]

(b) Let $a=2$ and $b=0.5$. A1

$e^{\frac{a}{b}}=e^{\frac{2}{0.5}}$

$e^{\frac{a}{b}}=e^4$

$e^a+e^b=e^2+e^{0.5}$ M1

$e^a+e^b=e^{0.5}(e^{1.5}+1)$

As $e^{0.5}<e^2$ and $e^{1.5}+1<e^2$, $e^{0.5}(e^{1.5}+1)<e^2\cdot e^2$

$\therefore e^{\frac{a}{b}}>e^a+e^b$

Thus, $e^{\frac{a}{b}}\le e^a+e^b$ is not always true. AG

[2]

3. (a) Let $a=-1$ and $b=2$. A1

$\frac{1}{a^2}=\frac{1}{(-1)^2}$

$\frac{1}{a^2}=1$

$\frac{1}{b^2}=\frac{1}{2^2}$ M1

$\frac{1}{b^2}=\frac{1}{4}$

$\therefore \frac{1}{a^2}>\frac{1}{b^2}$

Thus, $\frac{1}{a^2}<\frac{1}{b^2}$ is not always true. AG

[2]

(b) Suppose $2^x+1,\ 2^{2x}+1,\ 2^{3x}+1,\ \ldots$ is an arithmetic sequence. M1

$(2^{2x}+1)-(2^x+1)=(2^{3x}+1)-(2^{2x}+1)$ A1

$2^{2x}+1-2^x-1=2^{3x}+1-2^{2x}-1$

$2^{2x}-2^x=2^{3x}-2^{2x}$

$2^x-1=2^{2x}-2^x$

$(2^x)^2-2(2^x)+1=0$ A1

$(2^x-1)^2=0$

$2^x-1=0$

$2^x=1$

$x=0$, which contradicts with the statement x is a positive real value.

Thus, the sequence $2^x+1,\ 2^{2x}+1,\ 2^{3x}+1,\ \ldots$ is not an arithmetic sequence for all positive real values of x. AG

[3]

4. (a) Let $x = 0.25$. A1

$\log_2 x = \log_2 0.25$

$\log_2 x = \log_2 2^{-2}$

$\log_2 x = -2$

$\log_4 x = \log_4 0.25$ M1

$\log_4 x = \log_4 4^{-1}$

$\log_4 x = -1$

$\therefore \log_2 x < \log_4 x$

Thus, $\log_2 x \geq \log_4 x$ is not always true. AG

[2]

(b) Suppose $\log x$, $\log x^2$, $\log x^3$, … is a geometric sequence. M1

$$\frac{\log x^2}{\log x} = \frac{\log x^3}{\log x^2}$$ A1

$$\frac{2\log x}{\log x} = \frac{3\log x}{2\log x}$$ A1

$2 = \frac{3}{2}$, which arrives at a contradiction.

Thus, the sequence $\log x$, $\log x^2$, $\log x^3$, … is not a geometric sequence for all positive real values of x. AG

[3]

Chapter 7 Solution

Exercise 20

1. $\cos 2\alpha = 1 - 2\sin^2 \alpha$

$\cos 2\alpha = 1 - 2\left(\frac{2}{3}\right)^2$ (A1) for substitution

$\cos 2\alpha = \frac{1}{9}$

$\sec 4\alpha = \frac{1}{\cos 2(2\alpha)}$ (M1) for valid approach

$\sec 4\alpha = \frac{1}{2\cos^2 2\alpha - 1}$

$\sec 4\alpha = \frac{1}{2\left(\frac{1}{9}\right)^2 - 1}$ A1

$\sec 4\alpha = -\frac{81}{79}$ A1

[4]

2. $\tan\alpha = \sqrt{\sec^2\alpha - 1}$ (M1) for valid approach

$$\tan\alpha = \sqrt{\frac{1}{\cos^2\alpha} - 1}$$

$$\tan\alpha = \sqrt{\frac{1}{\left(-\frac{3}{5}\right)^2} - 1}$$ (A1) for substitution

$$\tan\alpha = \frac{4}{3}$$

$$\tan 2\alpha = \frac{2\tan\alpha}{1 - \tan^2\alpha}$$

$$\tan 2\alpha = \frac{2\left(\frac{4}{3}\right)}{1 - \left(\frac{4}{3}\right)^2}$$ A1

$$\tan 2\alpha = \frac{\frac{8}{3}}{-\frac{7}{9}}$$

$$\tan 2\alpha = -\frac{24}{7}$$ A1

[4]

3. $\cos\alpha + \sin\alpha = \frac{\sqrt{3}}{2}$

$(\cos\alpha + \sin\alpha)^2 = \left(\frac{\sqrt{3}}{2}\right)^2$ (M1) for valid approach

$\cos^2\alpha + 2\sin\alpha\cos\alpha + \sin^2\alpha = \frac{3}{4}$

$1 + \sin 2\alpha = \frac{3}{4}$ A1

$\sin 2\alpha = -\frac{1}{4}$ (A1) for correct value

$\sec 4\alpha = \frac{1}{\cos 2(2\alpha)}$ (M1) for valid approach

$\sec 4\alpha = \frac{1}{1 - 2\sin^2 2\alpha}$

$\sec 4\alpha = \frac{1}{1 - 2\left(-\frac{1}{4}\right)^2}$ A1

$\sec 4\alpha = \frac{8}{7}$ A1

[6]

4. $(\sec\alpha+\tan\alpha)^2=\frac{3}{2}+2\sec\alpha\tan\alpha$

$\sec^2\alpha+2\sec\alpha\tan\alpha+\tan^2\alpha=\frac{3}{2}+2\sec\alpha\tan\alpha$ (M1) for valid approach

$\sec^2\alpha+\tan^2\alpha=\frac{3}{2}$

$1+\tan^2\alpha+\tan^2\alpha=\frac{3}{2}$ A1

$2\tan^2\alpha=\frac{1}{2}$

$\tan^2\alpha=\frac{1}{4}$

$\tan\alpha=\frac{1}{2}$ or $\tan\alpha=-\frac{1}{2}$ (*Rejected*) (A1) for correct value

$\cot 2\alpha=\frac{1-\tan^2\alpha}{2\tan\alpha}$ (M1) for valid approach

$\cot 2\alpha=\frac{1-\left(\frac{1}{2}\right)^2}{2\left(\frac{1}{2}\right)}$ A1

$\cot 2\alpha=\frac{3}{4}$ A1

[6]

Exercise 21

1. $\tan 2x = \tan x$

$\frac{2\tan x}{1-\tan^2 x} = \tan x$ — (A1) for substitution

$2\tan x = \tan x(1-\tan^2 x)$ — (M1) for valid approach

$\tan x(2-(1-\tan^2 x)) = 0$

$\tan x = 0$ or $\tan^2 x = -1$ (*Rejected*) — A1

$x = 0$, $x = \pi$, $x = 2\pi$, $x = 3\pi$ or $x = 4\pi$ — A2

[5]

2. $\operatorname{cosec}^2 x + 2\cot x = 0$

$\cot^2 x + 1 + 2\cot x = 0$ — (A1) for substitution

$\cot^2 x + 2\cot x + 1 = 0$

$(\cot x + 1)^2 = 0$ — (A1) for factorization

$\cot x = -1$

$\tan x = -1$ — A1

$x = \frac{3\pi}{4}$ — A1

[4]

3. $\sec^2 2x + \tan 2x = 1$

$\tan^2 2x + 1 + \tan 2x = 1$ — (A1) for substitution

$\tan^2 2x + \tan 2x = 0$

$\tan 2x(\tan 2x + 1) = 0$ — (A1) for factorization

$\tan 2x = 0$ or $\tan 2x = -1$ — A1

$2x = \pi$, $2x = 2\pi$ or $2x = \pi - \frac{\pi}{4}$, $2x = 2\pi - \frac{\pi}{4}$

$\therefore x = \frac{3\pi}{8}$ (*Rejected*), $x = \frac{\pi}{2}$, $x = \frac{7\pi}{8}$ or $x = \pi$ (*Rejected*) — A2

[5]

4. $\tan x + \cot x = 4$

$\dfrac{\sin x}{\cos x} + \dfrac{\cos x}{\sin x} = 4$ (A1) for substitution

$\dfrac{\sin^2 x + \cos^2 x}{\sin x \cos x} = 4$

$1 = 4\sin x \cos x$ (M1) for valid approach

$1 = 2(2\sin x \cos x)$

$\sin 2x = \dfrac{1}{2}$ A1

$2x = \dfrac{\pi}{6}$ or $2x = \pi - \dfrac{\pi}{6}$

$x = \dfrac{\pi}{12}$ or $x = \dfrac{11\pi}{12}$ A2

[5]

Exercise 22

1. $\cos\alpha = \sqrt{1-\sin^2\alpha}$

$\cos\alpha = \sqrt{1-\left(\frac{2}{5}\right)^2}$ (A1) for substitution

$\cos\alpha = \frac{\sqrt{21}}{5}$

$\sin\beta = \sqrt{1-\cos^2\beta}$

$\sin\beta = \sqrt{1-\left(\frac{3}{5}\right)^2}$ (A1) for substitution

$\sin\beta = \frac{4}{5}$

$\cos(\alpha-2\beta) = \cos\alpha\cos 2\beta + \sin\alpha\sin 2\beta$ A1

$\cos(\alpha-2\beta) = \cos\alpha(2\cos^2\beta-1) + \sin\alpha(2\sin\beta\cos\beta)$ A1

$\cos(\alpha-2\beta) = \left(\frac{\sqrt{21}}{5}\right)\left(2\left(\frac{3}{5}\right)^2-1\right) + 2\left(\frac{2}{5}\right)\left(\frac{4}{5}\right)\left(\frac{3}{5}\right)$ (A1) for substitution

$\cos(\alpha-2\beta) = \left(\frac{\sqrt{21}}{5}\right)\left(\frac{-7}{25}\right) + \frac{48}{125}$

$\cos(\alpha-2\beta) = \frac{-7\sqrt{21}+48}{125}$ A1

[6]

2. $\sec\alpha = \sqrt{1+\tan^2\alpha}$

$\sec\alpha = \sqrt{1+\left(\frac{3}{4}\right)^2}$ (A1) for substitution

$\sec\alpha = \frac{5}{4}$

$\therefore \cos\alpha = \frac{4}{5}$

$\sin\alpha = \sqrt{1-\cos^2\alpha}$

$\sin\alpha = \sqrt{1-\left(\frac{4}{5}\right)^2}$ (A1) for substitution

$\sin\alpha = \frac{3}{5}$

$\sin\beta = \sqrt{1-\cos^2\beta}$

$\sin\beta = \sqrt{1-\left(\frac{\sqrt{3}}{5}\right)^2}$ (A1) for substitution

$\sin\beta = \frac{\sqrt{22}}{5}$

$\sin(\alpha+2\beta) = \sin\alpha\cos2\beta + \cos\alpha\sin2\beta$ A1

$\sin(\alpha+2\beta) = \sin\alpha(2\cos^2\beta - 1) + \cos\alpha(2\sin\beta\cos\beta)$ A1

$\sin(\alpha+2\beta) = \left(\frac{3}{5}\right)\left(2\left(\frac{\sqrt{3}}{5}\right)^2 - 1\right) + 2\left(\frac{4}{5}\right)\left(\frac{\sqrt{22}}{5}\right)\left(\frac{\sqrt{3}}{5}\right)$ (A1) for substitution

$\sin(\alpha+2\beta) = \left(\frac{3}{5}\right)\left(\frac{-19}{25}\right) + \frac{8\sqrt{66}}{125}$

$\sin(\alpha+2\beta) = \frac{-57+8\sqrt{66}}{125}$ A1

[7]

3. $\cos\left(x+\frac{\pi}{6}\right)=\cos\frac{\pi}{3}\cos x$

$\cos x\cos\frac{\pi}{6}-\sin x\sin\frac{\pi}{6}=\cos\frac{\pi}{3}\cos x$ (A1) for substitution

$\frac{\sqrt{3}}{2}\cos x-\frac{1}{2}\sin x=\frac{1}{2}\cos x$ (A1) for correct values

$\sqrt{3}\cos x-\sin x=\cos x$

$\sqrt{3}\cos x-\cos x=\sin x$ A1

$\sin x=(\sqrt{3}-1)\cos x$

$\tan x=\sqrt{3}-1$

$\therefore a=1,\ b=-1$ A2

[5]

4. $\tan\left(x+\frac{\pi}{4}\right)=2$

$\frac{\tan x+\tan\frac{\pi}{4}}{1-\tan x\tan\frac{\pi}{4}}=2$ (A1) for substitution

$\frac{\tan x+1}{1-\tan x}=2$ (A1) for correct value

$\tan x+1=2(1-\tan x)$

$\tan x+1=2-2\tan x$

$3\tan x=1$

$\tan x=\frac{1}{3}$ A1

$\sec^2 x=1+\tan^2 x$ A1

$\sec^2 x=1+\left(\frac{1}{3}\right)^2$

$\sec^2 x=\frac{10}{9}$ A1

[5]

Exercise 23

1. (a) $1-\tan x \geq 0$ A1

$\tan x \leq 1$

$\tan\left(-\frac{\pi}{2}\right) < \tan x \leq \tan\frac{\pi}{4}$

$-\frac{\pi}{2} < x \leq \frac{\pi}{4}$

Thus, the largest possible domain of f is

$\left\{x : -\frac{\pi}{2} < x \leq \frac{\pi}{4}\right\}$. A1

[2]

(b) $y = \sqrt{1-\tan x}$

$\Rightarrow x = \sqrt{1-\tan y}$ (M1) for swapping variables

$x^2 = 1-\tan y$ M1

$\tan y = 1-x^2$

$y = \arctan(1-x^2)$

$\therefore f^{-1}(x) = \arctan(1-x^2)$ A1

[3]

2. (a) $\arcsin x - \frac{\pi}{6} \geq 0$ A1

$\arcsin x \geq \frac{\pi}{6}$

$\frac{\pi}{6} \leq \arcsin x \leq \frac{\pi}{2}$

$\sin\frac{\pi}{6} \leq x \leq \sin\frac{\pi}{2}$ A1

$\frac{1}{2} \leq x \leq 1$

Thus, the largest possible domain of f is

$\left\{x : \frac{1}{2} \leq x \leq 1\right\}$. A1

[3]

(b) $y = 2\sqrt{\arcsin x - \frac{\pi}{6}}$

$\Rightarrow x = 2\sqrt{\arcsin y - \frac{\pi}{6}}$ (M1) for swapping variables

$\Rightarrow \frac{x}{2} = \sqrt{\arcsin y - \frac{\pi}{6}}$

$\frac{x^2}{4} = \arcsin y - \frac{\pi}{6}$ M1

$\arcsin y = \frac{x^2}{4} + \frac{\pi}{6}$

$y = \sin\left(\frac{x^2}{4} + \frac{\pi}{6}\right)$

$\therefore f^{-1}(x) = \sin\left(\frac{x^2}{4} + \frac{\pi}{6}\right)$ A1

[3]

3. (a) Let $A = \arctan\frac{1}{3}$ and $B = \arctan\frac{1}{7}$.

$A + B = \arctan\frac{1}{m}$

$\tan(A+B) = \dfrac{\tan A + \tan B}{1 - \tan A \tan B}$ (M1) for valid approach

$\dfrac{1}{m} = \dfrac{\frac{1}{3}+\frac{1}{7}}{1-\left(\frac{1}{3}\right)\left(\frac{1}{7}\right)}$ (A1) for substitution

$\dfrac{1}{m} = \dfrac{\frac{10}{21}}{\frac{20}{21}}$

$\dfrac{1}{m} = \dfrac{1}{2}$

$\therefore m = 2$ A1

[3]

(b) $\tan(2A+B) = \tan(A+A+B)$

$\tan(2A+B) = \dfrac{\tan A + \tan(A+B)}{1 - \tan A \tan(A+B)}$ (M1) for valid approach

$\tan(2A+B) = \dfrac{\frac{1}{3}+\frac{1}{2}}{1-\left(\frac{1}{3}\right)\left(\frac{1}{2}\right)}$ (A1) for substitution

$\tan(2A+B) = \dfrac{\frac{5}{6}}{\frac{5}{6}}$

$\tan(2A+B) = 1$

$2A + B = \dfrac{\pi}{4}$

$\therefore 2\arctan\frac{1}{3} + \arctan\frac{1}{7} = \frac{\pi}{4}$ A1

[3]

4. (a) Let $A = \arctan\frac{1}{5}$ and $B = \arctan\frac{1}{7}$.

$A - B = \arctan\frac{1}{r}$

$\tan(A-B) = \frac{\tan A - \tan B}{1 + \tan A \tan B}$ (M1) for valid approach

$\frac{1}{r} = \frac{\frac{1}{5} - \frac{1}{7}}{1 + \left(\frac{1}{5}\right)\left(\frac{1}{7}\right)}$ (A1) for substitution

$\frac{1}{r} = \frac{\frac{2}{35}}{\frac{36}{35}}$

$\frac{1}{r} = \frac{1}{18}$

$\therefore r = 18$ A1

[3]

(b) $\arctan 5 - \arctan 7 = \left(\frac{\pi}{2} - \arctan\frac{1}{5}\right) - \left(\frac{\pi}{2} - \arctan\frac{1}{7}\right)$ (M1) for valid approach

$\arctan 5 - \arctan 7 = -\left(\arctan\frac{1}{5} - \arctan\frac{1}{7}\right)$

$\arctan 5 - \arctan 7 = -\arctan\frac{1}{18}$ A1

[2]

Exercise 24

1. (a) $f(-x) = f(x)$

$a(-x) + b\cos(-x) = ax + b\cos x$ (M1) for valid approach

$-ax + b\cos x = ax + b\cos x$

$-ax = ax$

$0 = 2ax$

$a = 0$ A1

$b \in \mathbb{R}$ A1

[3]

(b) $h(-x) = \dfrac{f(-x)}{g(-x)}$ M1

$h(-x) = \dfrac{f(x)}{-(-x)^5}$

$h(-x) = -\dfrac{f(x)}{-x^5}$ A1

$h(-x) = -\dfrac{f(x)}{g(x)}$

$h(-x) = -h(x)$

Thus, h is an odd function. AG

[2]

2. (a) $f(-x) = f(x)$

$a\sec(-x) - \dfrac{b}{-x} = a\sec x - \dfrac{b}{x}$ (M1) for valid approach

$a\sec x + \dfrac{b}{x} = a\sec x - \dfrac{b}{x}$

$a \in \mathbb{R}$ A1

$\dfrac{b}{x} = -\dfrac{b}{x}$

$\dfrac{2b}{x} = 0$

$b = 0$ A1

[3]

(b) $h(-x) = f(-x) + g(-x)$ M1

$h(-x) = f(x) + \sin 4(-x)^2$

$h(-x) = f(x) + \sin 4x^2$ A1

$h(-x) = f(x) + g(x)$

$h(-x) = h(x)$

Thus, h is an even function. AG

[2]

3. (a) $f(-x)=\left|\sin\frac{-x}{2}\right|$ M1

$f(-x)=\left|-\sin\frac{x}{2}\right|$ A1

$f(-x)=|-1|\left|\sin\frac{x}{2}\right|$

$f(-x)=\left|\sin\frac{x}{2}\right|$

$f(-x)=f(x)$

Thus, f is an even function. AG

[2]

(b) $h(x)=g(f(x))$

$h(x)=\frac{1}{1+(f(x))^2}$ (M1) for valid approach

$h(-p)=\frac{1}{1+(f(-p))^2}$

$h(-p)=\frac{1}{1+(f(p))^2}$

$h(-p)=h(p)$

$\therefore h(-p)=q$ A1

[2]

4. (a) $f(-x)=(-x)^2\operatorname{cosec}2(-x)$ M1

$f(-x)=x^2(-\operatorname{cosec}2x)$ A1

$f(-x)=-x^2\operatorname{cosec}2x$

$f(-x)=-f(x)$

Thus, f is an odd function. AG

[2]

(b) $h(x)=g(f(x))$

$h(x)=\tan f(x)$ (M1) for valid approach

$h(1.1)=\tan f(1.1)$

$h(1.1)=\tan(-f(-1.1))$

$h(1.1)=-\tan f(-1.1)$ A1

$h(1.1)=-g(f(-1.1))$

$h(1.1)=-h(-1.1)$

$\therefore h(1.1)=-r$ A1

[3]

Exercise 25

1. (a) $\frac{2\pi}{B} = 2\left(\frac{3\pi}{4} - \frac{\pi}{4}\right)$

$\frac{2\pi}{B} = \pi$

$B = 2$ A1

$3 = A\text{cosec}2\left(\frac{\pi}{4}\right) + C$

$3 = A + C$

$C = 3 - A$

$\frac{7}{3} = A\text{cosec}2\left(\frac{3\pi}{4}\right) + C$

$\therefore \frac{7}{3} = -A + 3 - A$ (M1) for substitution

$-\frac{2}{3} = -2A$

$A = \frac{1}{3}$ A1

$C = 3 - \frac{1}{3}$

$C = \frac{8}{3}$ A1

[4]

(b) $\left\{y : y \le \frac{7}{3} \text{ or } y \ge 3\right\}$ A1

[1]

2. (a) $B = \frac{\pi}{6} - 0$

$B = \frac{\pi}{6}$ A1

$-1 = A\sec\left(\frac{\pi}{6} - \frac{\pi}{6}\right) + C$

$-1 = A + C$

$C = -1 - A$

$2 = A\sec\left(\frac{\pi}{2} - \frac{\pi}{6}\right) + C$

$\therefore 2 = 2A + (-1 - A)$ (M1) for substitution

$A = 3$ A1

$C = -1 - 3$

$C = -4$ A1

[4]

(b) $\{y : y \le -7 \text{ or } y \ge -1\}$ A1

[1]

3. (a) $\frac{\pi}{B} = \frac{3}{4} - \frac{1}{4}$ (M1) for valid approach

$\frac{\pi}{B} = \frac{1}{2}$

$B = 2\pi$ A1

$0 = A\cot 2\pi\left(\frac{1}{4}\right) + C$

$C = 0$ A1

$\frac{1}{4} = A\cot 2\pi\left(\frac{1}{8}\right) + 0$

$A = \frac{1}{4}$ A1

[4]

(b) $x = \frac{1}{2}$, $x = 1$ and $x = \frac{3}{2}$ A2

[2]

4. (a) $\frac{2\pi}{B}=2\left(\frac{5}{2}-\frac{3}{2}\right)$

$\frac{2\pi}{B}=2$

$B=\pi$ A1

$\pi-4=A\text{cosec}\pi\left(\frac{3}{2}\right)+C$

$\pi-4=-A+C$

$C=\pi-4+A$

$\pi+4=A\text{cosec}\pi\left(\frac{5}{2}\right)+C$

$\therefore \pi+4=A+\pi-4+A$ (M1) for substitution

$8=2A$

$A=4$ A1

$C=\pi-4+4$

$C=\pi$ A1

[4]

(b) $f(x)=8+\pi$

$4\text{cosec}\pi x+\pi=8+\pi$ (A1) for setting equation

$4\text{cosec}\pi x=8$

$\text{cosec}\pi x=2$

$\sin\pi x=\frac{1}{2}$ A1

$\pi x=\frac{\pi}{6},\ \pi x=\pi-\frac{\pi}{6},\ \pi x=\frac{\pi}{6}+2\pi$

or $\pi x=\pi-\frac{\pi}{6}+2\pi$

$x=\frac{1}{6},\ x=\frac{5}{6},\ x=\frac{13}{6}$ or $x=\frac{17}{6}$ A2

[4]

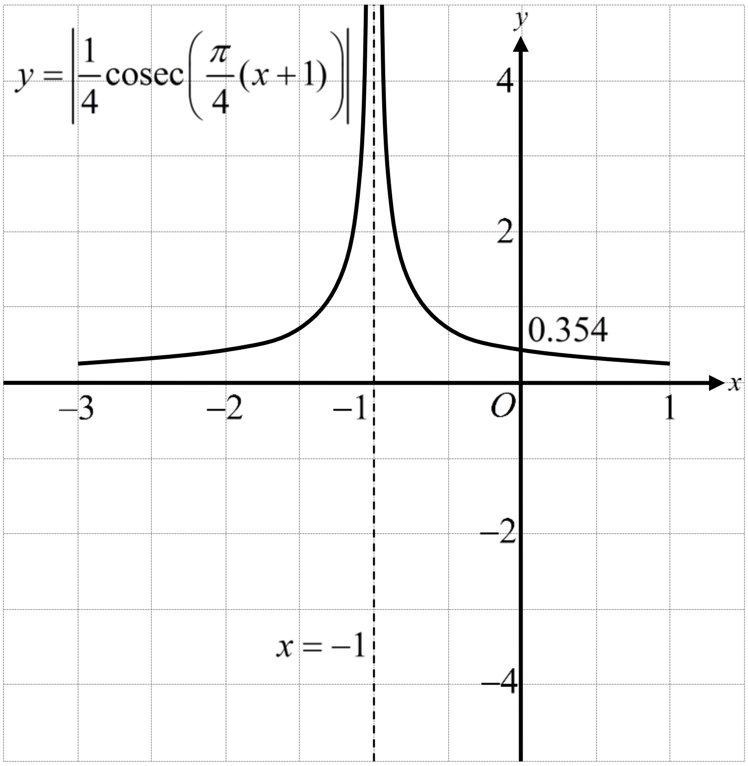
Exercise 26

1. (a) For correct shape A1

For correct asymptote A1

For correct intercept A1

[3]



(b) $\left|\frac{1}{4}\text{cosec}\left(\frac{\pi}{4}(x+1)\right)\right|+x=0$

By considering the graph of

$y=\left|\frac{1}{4}\text{cosec}\left(\frac{\pi}{4}(x+1)\right)\right|+x$, $x=-1.255283$. (M1) for valid approach

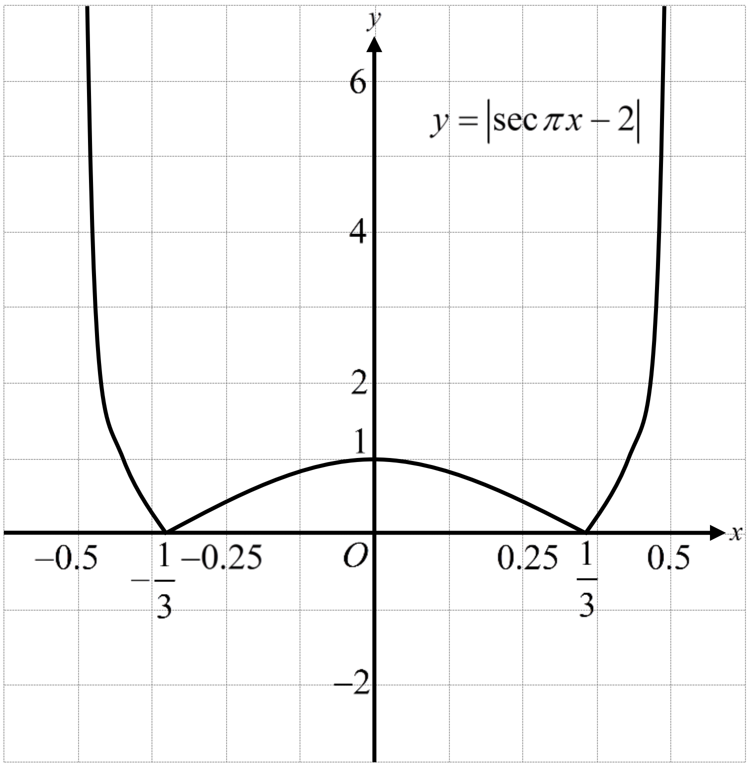
$\therefore x=-1.26$ A1

[2]

2. (a) For correct shape A1

For correct intercepts A2

[3]



(b) $|\sec\pi x-2|\geq 1$

$|\sec\pi x-2|-1\geq 0$

By considering the graph of $y=|\sec\pi x-2|-1$,

$x\leq -0.391826552$, $x=0$ or $x\geq 0.391826552$. (M1) for valid approach

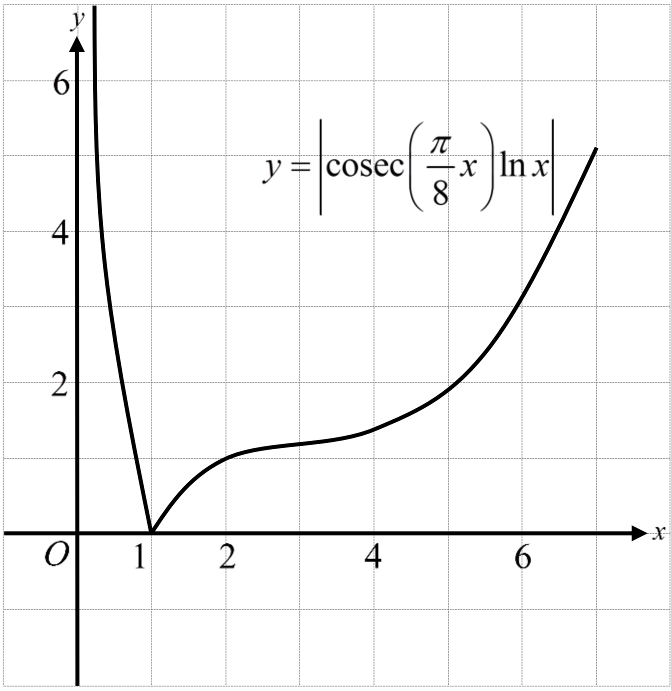
$\therefore -0.5<x\leq -0.392$, $x=0$ or $0.392\leq x<0.5$ A2

[3]

3. (a) For correct shape A2

For correct intercept A1

[3]



(b) $\left|\operatorname{cosec}\left(\frac{\pi}{8}x\right)\ln x\right| + x \geq 4$

$\left|\operatorname{cosec}\left(\frac{\pi}{8}x\right)\ln x\right| + x - 4 \geq 0$

By considering the graph of

$y = \left|\operatorname{cosec}\left(\frac{\pi}{8}x\right)\ln x\right| + x - 4$, $x \leq 0.5032849$ or

$x \geq 2.8380063$. (M1) for valid approach

$\therefore 0 < x \leq 0.503$ or $2.84 \leq x \leq 7$ A2

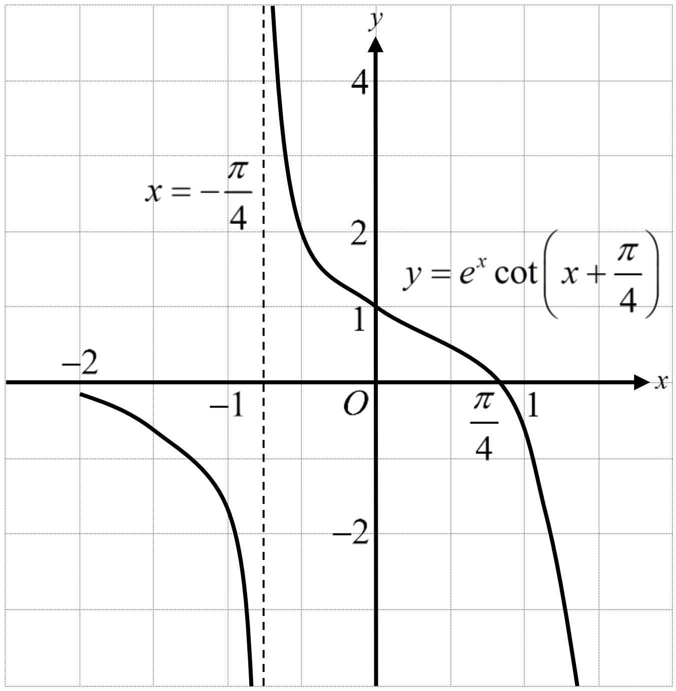
[3]

4. (a) For correct shape A1

For correct asymptote A1

For correct intercepts A1

[3]



(b) $k > -0.0503534375$

$k > -0.0504$ A2

[2]

Chapter 8 Solution

Exercise 27

1. $\because L_1$ and L_2 are perpendicular.

$\therefore \begin{pmatrix} k-1 \\ 20 \\ -10 \end{pmatrix} \cdot \begin{pmatrix} k+2 \\ k-2 \\ k \end{pmatrix} = 0$ (M1) for setting equation

$(k-1)(k+2)+(20)(k-2)+(-10)(k)=0$ (A1) for correct approach

$k^2+k-2+20k-40-10k=0$

$k^2+11k-42=0$

$(k+14)(k-3)=0$

$k=-14$ or $k=3$ A2

[4]

2. $\because$ The angle between L_1 and L_2 is not perpendicular.

$\therefore \begin{pmatrix} -4k \\ k \\ 0 \end{pmatrix} \cdot \begin{pmatrix} k \\ 1 \\ 1 \end{pmatrix} \neq 0$ (M1) for setting equation

$(-4k)(k)+(k)(1)+(0)(1)\neq 0$ (A1) for correct approach

$-4k^2+k\neq 0$

$4k^2-k\neq 0$

$k(4k-1)\neq 0$

$\therefore k\neq 0$ and $k\neq \dfrac{1}{4}$ A2

[4]

3. $L_1:\begin{cases} x=7+4s \\ y=5+3s \\ z=-s \end{cases}$, $L_2:\begin{cases} x=15+6t \\ y=-8-5t \\ z=1 \end{cases}$ (M1) for valid approach

$-s=1$ (A1) for correct approach

$s=-1$

$\therefore \begin{cases} x=7+4(-1)=3 \\ y=5+3(-1)=2 \\ z=-(-1)=1 \end{cases}$ (M1) for substitution

Thus, the coordinates of P are $(3, 2, 1)$. A2

[5]

4. $L_1:\begin{cases} x=k+3s \\ y=-5-4s \\ z=-4-3s \end{cases}$, $L_2:\begin{cases} x=5+t \\ y=3+2t \\ z=2+t \end{cases}$ (M1) for valid approach

$-4-3s=2+t$

$t=-6-3s$

$-5-4s=3+2t$

$\therefore -5-4s=3+2(-6-3s)$ (M1) for substitution

$-5-4s=-9-6s$

$2s=-4$

$s=-2$ A1

$t=-6-3(-2)$

$t=0$ A1

$k+3s=5+t$

$\therefore k+3(-2)=5+0$

$k=11$ A1

[5]

Exercise 28

1.	(a)	$\overrightarrow{RP} = \overrightarrow{OP} - \overrightarrow{OR}$	(M1) for valid approach
		$\overrightarrow{RP} = \overrightarrow{OP} - (\overrightarrow{OQ} + \overrightarrow{QR})$	
		$\overrightarrow{RP} = \overrightarrow{OP} - (\overrightarrow{OQ} + \frac{3}{2}\overrightarrow{OQ})$	(A1) for correct approach
		$\overrightarrow{RP} = \overrightarrow{OP} - \frac{5}{2}\overrightarrow{OQ}$	
		$\overrightarrow{RP} = \mathbf{p} - \frac{5}{2}\mathbf{q}$	A1
			[3]
	(b)	$\overrightarrow{RT} = \overrightarrow{RP} + \overrightarrow{PT}$	(M1) for valid approach
		$\overrightarrow{RT} = \overrightarrow{RP} + \frac{4}{5}\overrightarrow{PQ}$	
		$\overrightarrow{RT} = \overrightarrow{RP} + \frac{4}{5}(\overrightarrow{OQ} - \overrightarrow{OP})$	
		$\overrightarrow{RT} = \overrightarrow{RP} + \frac{4}{5}\overrightarrow{OQ} - \frac{4}{5}\overrightarrow{OP}$	(A1) for correct approach
		$\overrightarrow{RT} = \mathbf{p} - \frac{5}{2}\mathbf{q} + \frac{4}{5}\mathbf{q} - \frac{4}{5}\mathbf{p}$	(A1) for substitution
		$\overrightarrow{RT} = \frac{1}{5}\mathbf{p} - \frac{17}{10}\mathbf{q}$	A1
			[4]
	(c)	$\overrightarrow{TS} = \overrightarrow{RS} - \overrightarrow{RT}$	(M1) for valid approach
		$\overrightarrow{TS} = \overrightarrow{OS} - \overrightarrow{OR} - \overrightarrow{RT}$	
		$\overrightarrow{TS} = \frac{3}{5}\overrightarrow{OP} - \frac{5}{2}\overrightarrow{OQ} - \overrightarrow{RT}$	(A1) for correct approach
		$\overrightarrow{TS} = \frac{3}{5}\mathbf{p} - \frac{5}{2}\mathbf{q} - \left(\frac{1}{5}\mathbf{p} - \frac{17}{10}\mathbf{q}\right)$	(A1) for substitution
		$\overrightarrow{TS} = \frac{3}{5}\mathbf{p} - \frac{5}{2}\mathbf{q} - \frac{1}{5}\mathbf{p} + \frac{17}{10}\mathbf{q}$	
		$\overrightarrow{TS} = \frac{2}{5}\mathbf{p} - \frac{4}{5}\mathbf{q}$	A1
			[4]

(d) $\overrightarrow{TS} - \lambda\overrightarrow{RT} = 1.2\mu\mathbf{q}$

$\therefore \frac{2}{5}\mathbf{p} - \frac{4}{5}\mathbf{q} - \lambda\left(\frac{1}{5}\mathbf{p} - \frac{17}{10}\mathbf{q}\right) = 1.2\mu\mathbf{q}$ (M1) for substitution

$\frac{2}{5}\mathbf{p} - \frac{4}{5}\mathbf{q} - \frac{1}{5}\lambda\mathbf{p} + \frac{17}{10}\lambda\mathbf{q} = 1.2\mu\mathbf{q}$

$\left(\frac{2}{5} - \frac{1}{5}\lambda\right)\mathbf{p} + \left(-\frac{4}{5} + \frac{17}{10}\lambda\right)\mathbf{q} = 1.2\mu\mathbf{q}$ (A1) for simplification

$\frac{2}{5} - \frac{1}{5}\lambda = 0$

$2 - \lambda = 0$

$\lambda = 2$ A1

$-\frac{4}{5} + \frac{17}{10}(2) = 1.2\mu$ (M1) for substitution

$2.6 = 1.2\mu$

$\mu = \frac{13}{6}$ A1

[5]

2. (a) Let $PS:SQ = a:b$.

$\vec{OS} = \vec{OP} + \vec{PS}$	M1
$\vec{OS} = \vec{OP} + \frac{a}{a+b}\vec{PQ}$	A1
$\vec{OS} = \vec{OP} + \frac{a}{a+b}(\vec{OQ} - \vec{OP})$	
$\vec{OS} = \vec{OP} + \frac{a}{a+b}\vec{OQ} - \frac{a}{a+b}\vec{OP}$	
$\vec{OS} = \frac{b}{a+b}\vec{OP} + \frac{a}{a+b}\vec{OQ}$	A1
$\therefore \alpha + \beta = \frac{b}{a+b} + \frac{a}{a+b}$	M1
$\therefore \alpha + \beta = \frac{b+a}{a+b}$	
$\alpha + \beta = 1$	AG

[4]

(b)

$\vec{OH} = \vec{OP} + \vec{PH}$	(M1) for valid approach
$\vec{OH} = \vec{OP} + \frac{3}{5}\vec{PR}$	(A1) for correct approach
$\vec{OH} = \vec{OP} + \frac{3}{5}(\vec{OR} - \vec{OP})$	
$\vec{OH} = \vec{OP} + \frac{3}{5}\left(\frac{1}{4}\vec{OQ} - \vec{OP}\right)$	(A1) for correct approach
$\vec{OH} = \mathbf{p} + \frac{3}{5}\left(\frac{1}{4}\mathbf{q} - \mathbf{p}\right)$	(A1) for substitution
$\vec{OH} = \mathbf{p} + \frac{3}{20}\mathbf{q} - \frac{3}{5}\mathbf{p}$	
$\vec{OH} = \frac{2}{5}\mathbf{p} + \frac{3}{20}\mathbf{q}$	A1

[5]

(c) Let $\vec{OS} = c\vec{OH}$, $c \neq 0$.

$\vec{OS} = c\left(\frac{2}{5}\mathbf{p} + \frac{3}{20}\mathbf{q}\right)$

$\vec{OS} = \frac{2}{5}c\mathbf{p} + \frac{3}{20}c\mathbf{q}$ (M1) for valid approach

$\therefore \frac{2}{5}c + \frac{3}{20}c = 1$ A1

$\frac{11}{20}c = 1$

$c = \frac{20}{11}$ A1

$\therefore \vec{OS} = \frac{2}{5}\left(\frac{20}{11}\right)\mathbf{p} + \frac{3}{20}\left(\frac{20}{11}\right)\mathbf{q}$

$\vec{OS} = \frac{8}{11}\mathbf{p} + \frac{3}{11}\mathbf{q}$ A1

[4]

(d) $\vec{RS} = \vec{OS} - \vec{OR}$ M1

$\vec{RS} = \vec{OS} - \frac{1}{4}\vec{OQ}$

$\vec{RS} = \frac{8}{11}\mathbf{p} + \frac{3}{11}\mathbf{q} - \frac{1}{4}\mathbf{q}$ A1

$\vec{RS} = \frac{8}{11}\mathbf{p} + \frac{1}{44}\mathbf{q}$

Therefore, $\vec{RS}$ is not a multiple of $\mathbf{p}$. R1

Thus, RS and OP are not parallel. AG

[3]

3. (a) $\mathbf{p}\cdot\mathbf{q}=|\mathbf{p}||\mathbf{q}|\cos \mathrm{P\hat{O}Q}$ (M1) for valid approach

$24=(8)(6)\cos \mathrm{P\hat{O}Q}$ (A1) for substitution

$\cos \mathrm{P\hat{O}Q}=\frac{1}{2}$

$\mathrm{P\hat{O}Q}=60^\circ$ A1

[3]

(b) $\vec{OR}\cdot\vec{PQ}=\vec{OR}\cdot(\vec{OQ}-\vec{OP})$ M1

$\vec{OR}\cdot\vec{PQ}=\left(\frac{1}{6}\mathbf{p}+\frac{5}{9}\mathbf{q}\right)\cdot(\mathbf{q}-\mathbf{p})$

$\vec{OR}\cdot\vec{PQ}=\frac{1}{6}\mathbf{p}\cdot\mathbf{q}-\frac{1}{6}\mathbf{p}\cdot\mathbf{p}+\frac{5}{9}\mathbf{q}\cdot\mathbf{q}-\frac{5}{9}\mathbf{q}\cdot\mathbf{p}$ A1

$\vec{OR}\cdot\vec{PQ}=\frac{1}{6}\mathbf{p}\cdot\mathbf{q}-\frac{1}{6}|\mathbf{p}|^2+\frac{5}{9}|\mathbf{q}|^2-\frac{5}{9}\mathbf{p}\cdot\mathbf{q}$ A1

$\vec{OR}\cdot\vec{PQ}=\frac{1}{6}(24)-\frac{1}{6}(8)^2+\frac{5}{9}(6)^2-\frac{5}{9}(24)$ A1

$\vec{OR}\cdot\vec{PQ}=4-\frac{32}{3}+20-\frac{40}{3}$

$\vec{OR}\cdot\vec{PQ}=0$

$\therefore \mathrm{OR}\perp\mathrm{PQ}$ AG

$\vec{PR}\cdot\vec{OQ}=(\vec{OR}-\vec{OP})\cdot\vec{OQ}$ M1

$\vec{PR}\cdot\vec{OQ}=\left(\frac{1}{6}\mathbf{p}+\frac{5}{9}\mathbf{q}-\mathbf{p}\right)\cdot\mathbf{q}$

$\vec{PR}\cdot\vec{OQ}=\left(-\frac{5}{6}\mathbf{p}+\frac{5}{9}\mathbf{q}\right)\cdot\mathbf{q}$

$\vec{PR}\cdot\vec{OQ}=-\frac{5}{6}\mathbf{p}\cdot\mathbf{q}+\frac{5}{9}\mathbf{q}\cdot\mathbf{q}$ A1

$\vec{PR}\cdot\vec{OQ}=-\frac{5}{6}\mathbf{p}\cdot\mathbf{q}+\frac{5}{9}|\mathbf{q}|^2$ A1

$\vec{PR}\cdot\vec{OQ}=-\frac{5}{6}(24)+\frac{5}{9}(6)^2$ A1

$\vec{PR}\cdot\vec{OQ}=-20+20$

$\vec{PR}\cdot\vec{OQ}=0$

$\therefore \mathrm{PR}\perp\mathrm{OQ}$ AG

[8]

(c) $\overrightarrow{SH} = \lambda \overrightarrow{OR}$

$\overrightarrow{SH} = \lambda\left(\frac{1}{6}\mathbf{p} + \frac{5}{9}\mathbf{q}\right)$

$\left|\overrightarrow{SH}\right| = \lambda\left|\frac{1}{6}\mathbf{p} + \frac{5}{9}\mathbf{q}\right|$ (M1) for setting equation

$\therefore \frac{\sqrt{39}}{3} = \lambda\sqrt{\left(\frac{1}{6}\mathbf{p} + \frac{5}{9}\mathbf{q}\right)\cdot\left(\frac{1}{6}\mathbf{p} + \frac{5}{9}\mathbf{q}\right)}$ (A1) for correct approach

$\frac{\sqrt{39}}{3} = \lambda\sqrt{\frac{1}{36}\mathbf{p}\cdot\mathbf{p} + \frac{5}{27}\mathbf{p}\cdot\mathbf{q} + \frac{25}{81}\mathbf{q}\cdot\mathbf{q}}$

$\frac{\sqrt{39}}{3} = \lambda\sqrt{\frac{1}{36}|\mathbf{p}|^2 + \frac{5}{27}\mathbf{p}\cdot\mathbf{q} + \frac{25}{81}|\mathbf{q}|^2}$

$\frac{\sqrt{39}}{3} = \lambda\sqrt{\frac{1}{36}(8)^2 + \frac{5}{27}(24) + \frac{25}{81}(6)^2}$ (A1) for substitution

$\frac{\sqrt{39}}{3} = \lambda\sqrt{\frac{52}{3}}$

$\frac{\sqrt{39}}{3} = \frac{\sqrt{156}}{3}\lambda$ M1

$\lambda = \frac{1}{2}$ A1

[5]

4. (a) $QU \perp OP$

$\therefore \overrightarrow{QU} \cdot \overrightarrow{OP} = 0$ M1

$(\overrightarrow{OU} - \overrightarrow{OQ}) \cdot \overrightarrow{OP} = 0$

$(\overrightarrow{OU} - \mathbf{q}) \cdot \mathbf{p} = 0$ A1

$\overrightarrow{OU} \cdot \mathbf{p} - \mathbf{q} \cdot \mathbf{p} = 0$

$\mathbf{p} \cdot \mathbf{q} = \overrightarrow{OU} \cdot \mathbf{p}$

$PU \perp OQ$

$\therefore \overrightarrow{PU} \cdot \overrightarrow{OQ} = 0$ M1

$(\overrightarrow{OU} - \overrightarrow{OP}) \cdot \overrightarrow{OQ} = 0$

$(\overrightarrow{OU} - \mathbf{p}) \cdot \mathbf{q} = 0$ A1

$\overrightarrow{OU} \cdot \mathbf{q} - \mathbf{p} \cdot \mathbf{q} = 0$

$\mathbf{p} \cdot \mathbf{q} = \overrightarrow{OU} \cdot \mathbf{q}$

$\therefore \mathbf{p} \cdot \mathbf{q} = \overrightarrow{OU} \cdot \mathbf{p} = \overrightarrow{OU} \cdot \mathbf{q}$ AG

[4]

(b) $\overrightarrow{OV} = \frac{2}{3}\overrightarrow{OH}$ (A1) for correct approach

$\overrightarrow{OV} = \frac{2}{3}(\overrightarrow{OP} + \overrightarrow{PH})$ (M1) for valid approach

$\overrightarrow{OV} = \frac{2}{3}\left(\overrightarrow{OP} + \frac{1}{2}\overrightarrow{PQ}\right)$ (A1) for correct approach

$\overrightarrow{OV} = \frac{2}{3}\left(\overrightarrow{OP} + \frac{1}{2}(\overrightarrow{OQ} - \overrightarrow{OP})\right)$

$\overrightarrow{OV} = \frac{2}{3}\left(\mathbf{p} + \frac{1}{2}(\mathbf{q} - \mathbf{p})\right)$ (A1) for substitution

$\overrightarrow{OV} = \frac{2}{3}\left(\frac{1}{2}\mathbf{p} + \frac{1}{2}\mathbf{q}\right)$

$\overrightarrow{OV} = \frac{1}{3}\mathbf{p} + \frac{1}{3}\mathbf{q}$ A1

[5]

(c) $\mathbf{q}\cdot(2\overrightarrow{WV}-\overrightarrow{VU})=\mathbf{q}\cdot(2(\overrightarrow{OV}-\overrightarrow{OW})-(\overrightarrow{OU}-\overrightarrow{OV}))$ M1A1

$\mathbf{q}\cdot(2\overrightarrow{WV}-\overrightarrow{VU})=\mathbf{q}\cdot(3\overrightarrow{OV}-2\overrightarrow{OW}-\overrightarrow{OU})$ A1

$\mathbf{q}\cdot(2\overrightarrow{WV}-\overrightarrow{VU})=3\mathbf{q}\cdot\overrightarrow{OV}-2\mathbf{q}\cdot\overrightarrow{OW}-\mathbf{q}\cdot\overrightarrow{OU}$ M1

$\mathbf{q}\cdot(2\overrightarrow{WV}-\overrightarrow{VU})=3\mathbf{q}\cdot\left(\frac{1}{3}\mathbf{p}+\frac{1}{3}\mathbf{q}\right)$

M1A1

$-2|\mathbf{q}|\left|\overrightarrow{OW}\right|\cos W\hat{O}K-\mathbf{p}\cdot\mathbf{q}$

$\mathbf{q}\cdot(2\overrightarrow{WV}-\overrightarrow{VU})=\mathbf{p}\cdot\mathbf{q}+|\mathbf{q}|^2-2|\mathbf{q}|\left(\frac{|\mathbf{q}|}{2}\right)-\mathbf{p}\cdot\mathbf{q}$ M1A1

$\mathbf{q}\cdot(2\overrightarrow{WV}-\overrightarrow{VU})=\mathbf{p}\cdot\mathbf{q}+|\mathbf{q}|^2-|\mathbf{q}|^2-\mathbf{p}\cdot\mathbf{q}$

$\mathbf{q}\cdot(2\overrightarrow{WV}-\overrightarrow{VU})=0$ AG

[8]

Exercise 29

1. (a) $\frac{1}{2}[(\mathbf{p}+\mathbf{q})\times(\mathbf{q}+\mathbf{r})]\cdot(\mathbf{r}+\mathbf{p})$

$=\frac{1}{2}[(\mathbf{p}+\mathbf{q})\times\mathbf{q}+(\mathbf{p}+\mathbf{q})\times\mathbf{r}]\cdot(\mathbf{r}+\mathbf{p})$ M1

$=\frac{1}{2}[\mathbf{p}\times\mathbf{q}+\mathbf{q}\times\mathbf{q}+\mathbf{p}\times\mathbf{r}+\mathbf{q}\times\mathbf{r}]\cdot(\mathbf{r}+\mathbf{p})$

$=\frac{1}{2}[\mathbf{p}\times\mathbf{q}+\mathbf{0}+\mathbf{p}\times\mathbf{r}+\mathbf{q}\times\mathbf{r}]\cdot(\mathbf{r}+\mathbf{p})$ A1

$=\frac{1}{2}[\mathbf{p}\times\mathbf{q}\cdot(\mathbf{r}+\mathbf{p})+\mathbf{p}\times\mathbf{r}\cdot(\mathbf{r}+\mathbf{p})+\mathbf{q}\times\mathbf{r}\cdot(\mathbf{r}+\mathbf{p})]$ M1

$=\frac{1}{2}[(\mathbf{p}\times\mathbf{q})\cdot\mathbf{r}+(\mathbf{p}\times\mathbf{q})\cdot\mathbf{p}+(\mathbf{p}\times\mathbf{r})\cdot\mathbf{r}$
$+(\mathbf{p}\times\mathbf{r})\cdot\mathbf{p}+(\mathbf{q}\times\mathbf{r})\cdot\mathbf{r}+(\mathbf{q}\times\mathbf{r})\cdot\mathbf{p}]$ M1

$=\frac{1}{2}[(\mathbf{p}\times\mathbf{q})\cdot\mathbf{r}+\mathbf{0}+\mathbf{0}+\mathbf{0}+\mathbf{0}+(\mathbf{q}\times\mathbf{r})\cdot\mathbf{p}]$ A1

$=\frac{1}{2}[(\mathbf{p}\times\mathbf{q})\cdot\mathbf{r}+(\mathbf{p}\times\mathbf{q})\cdot\mathbf{r}]$ A1

$=\frac{1}{2}[2(\mathbf{p}\times\mathbf{q})\cdot\mathbf{r}]$

$=(\mathbf{p}\times\mathbf{q})\cdot\mathbf{r}$ AG

[6]

(b) $\mathbf{p}\times(\mathbf{q}\times\mathbf{r})+(\mathbf{p}\cdot\mathbf{q})\mathbf{r}$

$$=\mathbf{p}\times((q_1\mathbf{i}+q_2\mathbf{j}+q_3\mathbf{k})\times(r_1\mathbf{i}+r_2\mathbf{j}+r_3\mathbf{k}))$$
$$+((p_1\mathbf{i}+p_2\mathbf{j}+p_3\mathbf{k})\cdot(q_1\mathbf{i}+q_2\mathbf{j}+q_3\mathbf{k}))\mathbf{r}$$ M1

$$=\begin{pmatrix}p_1\\p_2\\p_3\end{pmatrix}\times\begin{pmatrix}q_2r_3-q_3r_2\\q_3r_1-q_1r_3\\q_1r_2-q_2r_1\end{pmatrix}+(p_1q_1+p_2q_2+p_3q_3)\begin{pmatrix}r_1\\r_2\\r_3\end{pmatrix}$$ A2

$$=\begin{pmatrix}p_2(q_1r_2-q_2r_1)-p_3(q_3r_1-q_1r_3)\\p_3(q_2r_3-q_3r_2)-p_1(q_1r_2-q_2r_1)\\p_1(q_3r_1-q_1r_3)-p_2(q_2r_3-q_3r_2)\end{pmatrix}$$
$$+\begin{pmatrix}(p_1q_1+p_2q_2+p_3q_3)r_1\\(p_1q_1+p_2q_2+p_3q_3)r_2\\(p_1q_1+p_2q_2+p_3q_3)r_3\end{pmatrix}$$ A2

$$=\begin{pmatrix}p_2q_1r_2-p_2q_2r_1-p_3q_3r_1+p_3q_1r_3\\p_3q_2r_3-p_3q_3r_2-p_1q_1r_2+p_1q_2r_1\\p_1q_3r_1-p_1q_1r_3-p_2q_2r_3+p_2q_3r_2\end{pmatrix}$$
$$+\begin{pmatrix}p_1q_1r_1+p_2q_2r_1+p_3q_3r_1\\p_1q_1r_2+p_2q_2r_2+p_3q_3r_2\\p_1q_1r_3+p_2q_2r_3+p_3q_3r_3\end{pmatrix}$$ M1

$$=\begin{pmatrix}p_1q_1r_1+p_2q_1r_2+p_3q_1r_3\\p_1q_2r_1+p_2q_2r_2+p_3q_2r_3\\p_1q_3r_1+p_2q_3r_2+p_3q_3r_3\end{pmatrix}$$ A1

$$=\begin{pmatrix}(p_1r_1+p_2r_2+p_3r_3)q_1\\(p_1r_1+p_2r_2+p_3r_3)q_2\\(p_1r_1+p_2r_2+p_3r_3)q_3\end{pmatrix}$$

$$=(p_1r_1+p_2r_2+p_3r_3)\mathbf{q}$$ A1

$$=(\mathbf{p}\cdot\mathbf{r})\mathbf{q}$$ AG

[8]

(c) Similarly, $\mathbf{q}\times(\mathbf{r}\times\mathbf{p})+(\mathbf{q}\cdot\mathbf{r})\mathbf{p}=(\mathbf{q}\cdot\mathbf{p})\mathbf{r}$ and

$\mathbf{r}\times(\mathbf{p}\times\mathbf{q})+(\mathbf{r}\cdot\mathbf{p})\mathbf{q}=(\mathbf{r}\cdot\mathbf{q})\mathbf{p}$. A1

$\therefore \mathbf{p}\times(\mathbf{q}\times\mathbf{r})+\mathbf{q}\times(\mathbf{r}\times\mathbf{p})+\mathbf{r}\times(\mathbf{p}\times\mathbf{q})$

$=((\mathbf{p}\cdot\mathbf{r})\mathbf{q}-(\mathbf{p}\cdot\mathbf{q})\mathbf{r})+((\mathbf{q}\cdot\mathbf{p})\mathbf{r}-(\mathbf{q}\cdot\mathbf{r})\mathbf{p})$

$+((\mathbf{r}\cdot\mathbf{q})\mathbf{p}-(\mathbf{r}\cdot\mathbf{p})\mathbf{q})$ A1

$=(\mathbf{p}\cdot\mathbf{r})\mathbf{q}-(\mathbf{p}\cdot\mathbf{q})\mathbf{r}+(\mathbf{q}\cdot\mathbf{p})\mathbf{r}-(\mathbf{q}\cdot\mathbf{r})\mathbf{p}$

$+(\mathbf{r}\cdot\mathbf{q})\mathbf{p}-(\mathbf{r}\cdot\mathbf{p})\mathbf{q}$

$=(\mathbf{r}\cdot\mathbf{p})\mathbf{q}-(\mathbf{p}\cdot\mathbf{q})\mathbf{r}+(\mathbf{p}\cdot\mathbf{q})\mathbf{r}-(\mathbf{q}\cdot\mathbf{r})\mathbf{p}$

$+(\mathbf{q}\cdot\mathbf{r})\mathbf{p}-(\mathbf{r}\cdot\mathbf{p})\mathbf{q}$ A1

$=\mathbf{0}$ AG

[3]

2. (a)

$\mathbf{q}\times\mathbf{r}=\mathbf{q}\times(\mathbf{0}-\mathbf{p}-\mathbf{q})$	A1
$\mathbf{q}\times\mathbf{r}=\mathbf{q}\times(-\mathbf{p}-\mathbf{q})$	
$\mathbf{q}\times\mathbf{r}=\mathbf{q}\times(-\mathbf{p})-\mathbf{q}\times\mathbf{q}$	M1
$\mathbf{q}\times\mathbf{r}=-\mathbf{q}\times\mathbf{p}-\mathbf{q}\times\mathbf{q}$	
$\mathbf{q}\times\mathbf{r}=-(-\mathbf{p}\times\mathbf{q})-\mathbf{0}$	A1
$\mathbf{q}\times\mathbf{r}=\mathbf{p}\times\mathbf{q}$	
$\mathbf{r}\times\mathbf{p}=(\mathbf{0}-\mathbf{p}-\mathbf{q})\times\mathbf{p}$	A1
$\mathbf{r}\times\mathbf{p}=(-\mathbf{p}-\mathbf{q})\times\mathbf{p}$	
$\mathbf{r}\times\mathbf{p}=-\mathbf{p}\times\mathbf{p}-\mathbf{q}\times\mathbf{p}$	M1
$\mathbf{r}\times\mathbf{p}=\mathbf{0}-(-\mathbf{p}\times\mathbf{q})$	A1
$\mathbf{r}\times\mathbf{p}=\mathbf{p}\times\mathbf{q}$	
$\therefore\mathbf{p}\times\mathbf{q}=\mathbf{q}\times\mathbf{r}=\mathbf{r}\times\mathbf{p}$	AG

[6]

(b)

$\mathbf{p}=\lambda\mathbf{q}$	
$\mathbf{r}=\mathbf{0}-\mathbf{p}-\mathbf{q}$	
$\mathbf{r}=-\lambda\mathbf{q}-\mathbf{q}$	A1
$\therefore\mathbf{p}\times\mathbf{q}-\mathbf{q}\times\mathbf{r}-\mathbf{r}\times\mathbf{p}$	
$=\lambda\mathbf{q}\times\mathbf{q}-\mathbf{q}\times(-\lambda\mathbf{q}-\mathbf{q})-(-\lambda\mathbf{q}-\mathbf{q})\times\lambda\mathbf{q}$	M1
$=\lambda\mathbf{q}\times\mathbf{q}-\mathbf{q}\times(-\lambda\mathbf{q})+\mathbf{q}\times\mathbf{q}-(-\lambda\mathbf{q})\times\lambda\mathbf{q}+\mathbf{q}\times\lambda\mathbf{q}$	
$=\lambda\mathbf{q}\times\mathbf{q}+\lambda\mathbf{q}\times\mathbf{q}+\mathbf{q}\times\mathbf{q}+\lambda^2\mathbf{q}\times\mathbf{q}+\lambda\mathbf{q}\times\mathbf{q}$	A1
$=(\lambda^2+3\lambda+1)\mathbf{q}\times\mathbf{q}$	
$=(\lambda^2+3\lambda+1)\mathbf{0}$	
$=\mathbf{0}$	AG

[3]

(c)

$\mathbf{p}\times\mathbf{q}=\mathbf{q}\times\mathbf{r}=\mathbf{r}\times\mathbf{p}$	
$\lvert\mathbf{p}\times\mathbf{q}\rvert=\lvert\mathbf{q}\times\mathbf{r}\rvert=\lvert\mathbf{r}\times\mathbf{p}\rvert$	M1
$\lvert\mathbf{p}\rvert\lvert\mathbf{q}\rvert\sin \mathrm{Q\hat{R}P}=\lvert\mathbf{q}\rvert\lvert\mathbf{r}\rvert\sin \mathrm{R\hat{P}Q}=\lvert\mathbf{p}\rvert\lvert\mathbf{r}\rvert\sin \mathrm{P\hat{Q}R}$	A1
$\dfrac{\lvert\mathbf{p}\rvert\lvert\mathbf{q}\rvert\sin \mathrm{Q\hat{R}P}}{\lvert\mathbf{p}\rvert\lvert\mathbf{q}\rvert\lvert\mathbf{r}\rvert}=\dfrac{\lvert\mathbf{q}\rvert\lvert\mathbf{r}\rvert\sin \mathrm{R\hat{P}Q}}{\lvert\mathbf{p}\rvert\lvert\mathbf{q}\rvert\lvert\mathbf{r}\rvert}=\dfrac{\lvert\mathbf{p}\rvert\lvert\mathbf{r}\rvert\sin \mathrm{P\hat{Q}R}}{\lvert\mathbf{p}\rvert\lvert\mathbf{q}\rvert\lvert\mathbf{r}\rvert}$	M1
$\dfrac{\sin \mathrm{Q\hat{R}P}}{\lvert\mathbf{r}\rvert}=\dfrac{\sin \mathrm{R\hat{P}Q}}{\lvert\mathbf{p}\rvert}=\dfrac{\sin \mathrm{P\hat{Q}R}}{\lvert\mathbf{q}\rvert}$	
$\therefore\dfrac{\sin \mathrm{R\hat{P}Q}}{\lvert\mathbf{p}\rvert}=\dfrac{\sin \mathrm{P\hat{Q}R}}{\lvert\mathbf{q}\rvert}=\dfrac{\sin \mathrm{Q\hat{R}P}}{\lvert\mathbf{r}\rvert}$	AG

[3]

(d) $|-\mathbf{p}|^2 + |\mathbf{q}|^2 - 2|-\mathbf{p}||\mathbf{q}|\cos Q\hat{R}P$ A1

$= |-\mathbf{p}|^2 + |\mathbf{q}|^2 - 2(-\mathbf{p})\cdot\mathbf{q}$ A1

$= |\mathbf{p}|^2 + |\mathbf{q}|^2 + 2\mathbf{p}\cdot\mathbf{q}$

$= \mathbf{p}\cdot\mathbf{p} + \mathbf{p}\cdot\mathbf{q} + \mathbf{p}\cdot\mathbf{q} + \mathbf{q}\cdot\mathbf{q}$ M1

$= \mathbf{p}\cdot(\mathbf{p}+\mathbf{q}) + \mathbf{q}\cdot(\mathbf{p}+\mathbf{q})$

$= (\mathbf{p}+\mathbf{q})\cdot(\mathbf{p}+\mathbf{q})$ M1

$= |\mathbf{p}+\mathbf{q}|^2$

$= |-(\mathbf{p}+\mathbf{q})|^2$ M1

$= |\mathbf{r}|^2$ AG

[5]

3. (a) $\mathbf{r} = \lambda\mathbf{p} + \mathbf{q}$

$|\mathbf{r}|^2 = |\lambda\mathbf{p} + \mathbf{q}|^2$

$|\mathbf{r}|^2 = (\lambda\mathbf{p} + \mathbf{q}) \cdot (\lambda\mathbf{p} + \mathbf{q})$ M1

$|\mathbf{r}|^2 = \lambda\mathbf{p} \cdot \lambda\mathbf{p} + \lambda\mathbf{p} \cdot \mathbf{q} + \mathbf{q} \cdot \lambda\mathbf{p} + \mathbf{q} \cdot \mathbf{q}$ A1

$|\mathbf{r}|^2 = \lambda^2\mathbf{p} \cdot \mathbf{p} + \lambda\mathbf{p} \cdot \mathbf{q} + \lambda\mathbf{q} \cdot \mathbf{p} + \mathbf{q} \cdot \mathbf{q}$

$|\mathbf{r}|^2 = \lambda^2\mathbf{p} \cdot \mathbf{p} + \lambda\mathbf{p} \cdot \mathbf{q} + \lambda\mathbf{p} \cdot \mathbf{q} + \mathbf{q} \cdot \mathbf{q}$ M1

$|\mathbf{r}|^2 = \lambda^2 |\mathbf{p}|^2 + 2\lambda\mathbf{p} \cdot \mathbf{q} + |\mathbf{q}|^2$ AG

[3]

(b) $(\mathbf{p} + \mathbf{q} + \mathbf{r}) \cdot (\mathbf{p} \times (\mathbf{q} + \mathbf{r}))$

$= (\mathbf{p} + \mathbf{q} + \mathbf{r}) \cdot (\mathbf{p} \times \mathbf{q} + \mathbf{p} \times \mathbf{r})$ M1

$= \mathbf{p} \cdot (\mathbf{p} \times \mathbf{q}) + \mathbf{q} \cdot (\mathbf{p} \times \mathbf{q}) + \mathbf{r} \cdot (\mathbf{p} \times \mathbf{q})$
$+ \mathbf{p} \cdot (\mathbf{p} \times \mathbf{r}) + \mathbf{q} \cdot (\mathbf{p} \times \mathbf{r}) + \mathbf{r} \cdot (\mathbf{p} \times \mathbf{r})$ M1

$= \mathbf{0} + \mathbf{0} + \mathbf{r} \cdot (\mathbf{p} \times \mathbf{q}) + \mathbf{0} + \mathbf{q} \cdot (\mathbf{p} \times \mathbf{r}) + \mathbf{0}$ A1

$= (\mathbf{p} \times \mathbf{q}) \cdot \mathbf{r} - \mathbf{q} \cdot (\mathbf{r} \times \mathbf{p})$ M1

$= 0$ AG

[4]

(c) $\mathbf{v} \cdot (\mathbf{p} - \mathbf{v}) = \left(\frac{\mathbf{p} \cdot \mathbf{q}}{|\mathbf{q}|^2}\right)\mathbf{q} \cdot \left(\mathbf{p} - \left(\frac{\mathbf{p} \cdot \mathbf{q}}{|\mathbf{q}|^2}\right)\mathbf{q}\right)$

$\mathbf{v} \cdot (\mathbf{p} - \mathbf{v}) = \left(\frac{\mathbf{p} \cdot \mathbf{q}}{|\mathbf{q}|^2}\right)\mathbf{q} \cdot \mathbf{p} - \left(\frac{\mathbf{p} \cdot \mathbf{q}}{|\mathbf{q}|^2}\right)\mathbf{q} \cdot \left(\frac{\mathbf{p} \cdot \mathbf{q}}{|\mathbf{q}|^2}\right)\mathbf{q}$ M1

$\mathbf{v} \cdot (\mathbf{p} - \mathbf{v}) = \frac{(\mathbf{p} \cdot \mathbf{q})^2}{|\mathbf{q}|^2} - \frac{(\mathbf{p} \cdot \mathbf{q})^2}{|\mathbf{q}|^4}\mathbf{q} \cdot \mathbf{q}$ A1

$\mathbf{v} \cdot (\mathbf{p} - \mathbf{v}) = \frac{(\mathbf{p} \cdot \mathbf{q})^2}{|\mathbf{q}|^2} - \frac{(\mathbf{p} \cdot \mathbf{q})^2}{|\mathbf{q}|^4}|\mathbf{q}|^2$ A1

$\mathbf{v} \cdot (\mathbf{p} - \mathbf{v}) = \frac{(\mathbf{p} \cdot \mathbf{q})^2}{|\mathbf{q}|^2} - \frac{(\mathbf{p} \cdot \mathbf{q})^2}{|\mathbf{q}|^2}$

$\mathbf{v} \cdot (\mathbf{p} - \mathbf{v}) = 0$ AG

[3]

(d) $\because \mathbf{v}\cdot(\mathbf{p}-\mathbf{v})=\mathbf{0}$

$\therefore \cos\theta=\dfrac{|\mathbf{v}|}{|\mathbf{p}|}$ A1

$\sec\theta=\dfrac{|\mathbf{p}|}{|\mathbf{v}|}$ M1

$\sec^2\theta=\dfrac{|\mathbf{p}|^2}{|\mathbf{v}|^2}$ A1

$\sec^2\theta-1=\dfrac{|\mathbf{p}|^2}{|\mathbf{v}|^2}-1$ M1

$\therefore \tan^2\theta=\dfrac{|\mathbf{p}|^2}{|\mathbf{v}|^2}-1$ A1

$\tan^2\theta=\dfrac{|\mathbf{p}|^2}{|\mathbf{v}|^2}-\dfrac{|\mathbf{v}|^2}{|\mathbf{v}|^2}$

$\tan\theta=\sqrt{\dfrac{|\mathbf{p}|^2-|\mathbf{v}|^2}{|\mathbf{v}|^2}}$ AG

[5]

4. (a) $\overrightarrow{PN} = \overrightarrow{PQ} + \overrightarrow{QN}$

$\overrightarrow{PN} = \mathbf{r} + \frac{\lambda}{\lambda+1}\overrightarrow{QR}$ (A1) for correct approach

$\overrightarrow{PN} = \mathbf{r} + \frac{\lambda}{\lambda+1}(\overrightarrow{PR} - \overrightarrow{PQ})$ (M1) for valid approach

$\overrightarrow{PN} = \mathbf{r} + \frac{\lambda}{\lambda+1}(\mathbf{q} - \mathbf{r})$

$\overrightarrow{PN} = \frac{\lambda+1}{\lambda+1}\mathbf{r} + \frac{\lambda}{\lambda+1}\mathbf{q} - \frac{\lambda}{\lambda+1}\mathbf{r}$ (M1) for valid approach

$\overrightarrow{PN} = \frac{\lambda}{\lambda+1}\mathbf{q} + \frac{1}{\lambda+1}\mathbf{r}$ A1

[4]

(b) $\because PN \perp QR$

$\therefore \overrightarrow{PN} \cdot \overrightarrow{QR} = 0$ M1

$\left(\frac{\lambda}{\lambda+1}\mathbf{q} + \frac{1}{\lambda+1}\mathbf{r}\right) \cdot (\mathbf{q} - \mathbf{r}) = 0$ A1

$(\lambda\mathbf{q} + \mathbf{r}) \cdot (\mathbf{q} - \mathbf{r}) = 0$

$\lambda\mathbf{q} \cdot (\mathbf{q} - \mathbf{r}) + \mathbf{r} \cdot (\mathbf{q} - \mathbf{r}) = 0$ M1

$\lambda\mathbf{q} \cdot (\mathbf{q} - \mathbf{r}) = -\mathbf{r} \cdot (\mathbf{q} - \mathbf{r})$

$\lambda\mathbf{q} \cdot (\mathbf{q} - \mathbf{r}) = \mathbf{r} \cdot (\mathbf{r} - \mathbf{q})$ M1

$\lambda = \frac{\mathbf{r} \cdot (\mathbf{r} - \mathbf{q})}{\mathbf{q} \cdot (\mathbf{q} - \mathbf{r})}$ AG

[4]

(c) $\overrightarrow{PN} = \frac{\lambda}{\lambda+1}\mathbf{q} + \frac{1}{\lambda+1}\mathbf{r}$

$\overrightarrow{PN} = \frac{1}{\lambda+1}(\lambda\mathbf{q}+\mathbf{r})$

$\overrightarrow{PN} = \frac{1}{\frac{\mathbf{r}\cdot(\mathbf{r}-\mathbf{q})}{\mathbf{q}\cdot(\mathbf{q}-\mathbf{r})}+1}\left(\frac{\mathbf{r}\cdot(\mathbf{r}-\mathbf{q})}{\mathbf{q}\cdot(\mathbf{q}-\mathbf{r})}\mathbf{q}+\mathbf{r}\right)$ M1

$\overrightarrow{PN} = \frac{1}{\frac{\mathbf{r}\cdot(\mathbf{r}-\mathbf{q})}{\mathbf{q}\cdot(\mathbf{q}-\mathbf{r})}+\frac{\mathbf{q}\cdot(\mathbf{q}-\mathbf{r})}{\mathbf{q}\cdot(\mathbf{q}-\mathbf{r})}}\left(\frac{\mathbf{r}\cdot(\mathbf{r}-\mathbf{q})}{\mathbf{q}\cdot(\mathbf{q}-\mathbf{r})}\mathbf{q}+\frac{\mathbf{q}\cdot(\mathbf{q}-\mathbf{r})}{\mathbf{q}\cdot(\mathbf{q}-\mathbf{r})}\mathbf{r}\right)$ M1

$\overrightarrow{PN} = \frac{\mathbf{q}\cdot(\mathbf{q}-\mathbf{r})}{\mathbf{r}\cdot(\mathbf{r}-\mathbf{q})+\mathbf{q}\cdot(\mathbf{q}-\mathbf{r})}\left(\frac{\mathbf{r}\cdot(\mathbf{r}-\mathbf{q})}{\mathbf{q}\cdot(\mathbf{q}-\mathbf{r})}\mathbf{q}+\frac{\mathbf{q}\cdot(\mathbf{q}-\mathbf{r})}{\mathbf{q}\cdot(\mathbf{q}-\mathbf{r})}\mathbf{r}\right)$ A1

$\overrightarrow{PN} = \frac{(\mathbf{r}\cdot(\mathbf{r}-\mathbf{q}))\mathbf{q}+(\mathbf{q}\cdot(\mathbf{q}-\mathbf{r}))\mathbf{r}}{\mathbf{r}\cdot(\mathbf{r}-\mathbf{q})+\mathbf{q}\cdot(\mathbf{q}-\mathbf{r})}$ M1

$\overrightarrow{PN} = \frac{(\mathbf{r}\cdot(\mathbf{r}-\mathbf{q}))\mathbf{q}+(\mathbf{q}\cdot(\mathbf{q}-\mathbf{r}))\mathbf{r}}{\mathbf{r}\cdot(\mathbf{r}-\mathbf{q})-\mathbf{q}\cdot(\mathbf{r}-\mathbf{q})}$ A1

$\overrightarrow{PN} = \frac{(\mathbf{r}\cdot(\mathbf{r}-\mathbf{q}))\mathbf{q}+(\mathbf{q}\cdot(\mathbf{q}-\mathbf{r}))\mathbf{r}}{(\mathbf{r}-\mathbf{q})\cdot(\mathbf{r}-\mathbf{q})}$

$\overrightarrow{PN} = \frac{(\mathbf{r}\cdot(\mathbf{r}-\mathbf{q}))\mathbf{q}+(\mathbf{q}\cdot(\mathbf{q}-\mathbf{r}))\mathbf{r}}{|\mathbf{r}-\mathbf{q}|^2}$ A1

$|\mathbf{r}-\mathbf{q}|^2\,\overrightarrow{PN} = (\mathbf{r}\cdot(\mathbf{r}-\mathbf{q}))\mathbf{q}+(\mathbf{q}\cdot(\mathbf{q}-\mathbf{r}))\mathbf{r}$ M1

$(\mathbf{r}\cdot(\mathbf{r}-\mathbf{q}))\mathbf{q}+(\mathbf{q}\cdot(\mathbf{q}-\mathbf{r}))\mathbf{r}-|\mathbf{r}-\mathbf{q}|^2\,\overrightarrow{PN} = 0$ AG

[7]

Exercise 30

1. (a) $L_1:\begin{cases} x=15+6t \\ y=11+3t \\ z=6+2t \end{cases}$, $L_2:\begin{cases} x=-3s \\ y=7+2s \\ z=8+6s \end{cases}$ (M1) for valid approach

$15+6t=-3s$

$s=-5-2t$

$11+3t=7+2s$

$\therefore 11+3t=7+2(-5-2t)$ (M1) for substitution

$11+3t=-3-4t$

$7t=-14$

$t=-2$ A1

$\therefore \begin{cases} x=15+6(-2)=3 \\ y=11+3(-2)=5 \\ z=6+2(-2)=2 \end{cases}$ (M1) for substitution

Thus, the coordinates of C are $(3, 5, 2)$. A1

[5]

(b) The coordinates of A and B are $(15, 11, 6)$ and $(0, 7, 8)$ respectively. (A1) for correct values

$\overrightarrow{CA} = (15\mathbf{i}+11\mathbf{j}+6\mathbf{k})-(3\mathbf{i}+5\mathbf{j}+2\mathbf{k})$

$\overrightarrow{CA} = 12\mathbf{i}+6\mathbf{j}+4\mathbf{k}$ A1

$\overrightarrow{CB} = (7\mathbf{j}+8\mathbf{k})-(3\mathbf{i}+5\mathbf{j}+2\mathbf{k})$

$\overrightarrow{CB} = -3\mathbf{i}+2\mathbf{j}+6\mathbf{k}$ A1

The area of the triangle ABC

$= \frac{1}{2}\left|\overrightarrow{CA}\times\overrightarrow{CB}\right|$ (M1) for valid approach

$= \frac{1}{2}\left|(12\mathbf{i}+6\mathbf{j}+4\mathbf{k})\times(-3\mathbf{i}+2\mathbf{j}+6\mathbf{k})\right|$

$$= \frac{1}{2}\left|\begin{pmatrix} (6)(6)-(4)(2) \\ (4)(-3)-(12)(6) \\ (12)(2)-(6)(-3) \end{pmatrix}\right|$$

$= \frac{1}{2}\left|28\mathbf{i}-84\mathbf{j}+42\mathbf{k}\right|$

$= \frac{1}{2}(14)\left|2\mathbf{i}-6\mathbf{j}+3\mathbf{k}\right|$

$= 7\sqrt{2^2+(-6)^2+3^2}$ A1

$= 49$ A1

[6]

(c) The vector equation of the line L_3:

$$\mathbf{r} = \begin{pmatrix} 3 \\ 5 \\ 2 \end{pmatrix} + u\begin{pmatrix} 2 \\ -6 \\ 3 \end{pmatrix}$$ (A1) for correct values

$$\begin{cases} x = 3+2u \\ y = 5-6u \\ z = 2+3u \end{cases}$$ A1

[2]

(d) $3+2u = 73$

$2u = 70$

$u = 35$ (A1) for correct value

$d = 2+3(35)$

$d = 107$ A1

[2]

(e) $\vec{CD} = (73\mathbf{i} - 205\mathbf{j} + 107\mathbf{k}) - (3\mathbf{i} + 5\mathbf{j} + 2\mathbf{k})$

$\vec{CD} = 70\mathbf{i} - 210\mathbf{j} + 105\mathbf{k}$ A1

The volume of the pyramid ABCD

$= \frac{1}{3}(49)(\sqrt{70^2 + (-210)^2 + 105^2})$ M1A1

$= \frac{1}{3}(49)(35)(\sqrt{2^2 + (-6)^2 + 3^2})$

$= \frac{1}{3}(49)(35)(7)$ A1

$= \frac{12005}{3}$ AG

[4]

2. (a) $L_1:\begin{cases} x=8+2t \\ y=8 \\ z=7 \end{cases}$, $L_2:\begin{cases} x=6+s \\ y=8+2\sqrt{3}+\sqrt{3}s \\ z=7 \end{cases}$ M1

$8=8+2\sqrt{3}+\sqrt{3}s$

$-2\sqrt{3}=\sqrt{3}s$

$s=-2$ A1

$x=6+(-2)$ M1

$x=4$

Thus, the coordinates of C are $(4, 8, 7)$. AG

[3]

(b) $(\mathbf{i}+\sqrt{3}\mathbf{j})\cdot\mathbf{j}=|\mathbf{i}+\sqrt{3}\mathbf{j}||\mathbf{j}|\cos\theta$ (M1) for valid approach

$(1)(0)+(\sqrt{3})(1)=(\sqrt{1^2+(\sqrt{3})^2})(1)\cos\theta$ (A1) for correct approach

$\sqrt{3}=2\cos\theta$

$\cos\theta=\frac{\sqrt{3}}{2}$

$\theta=30^\circ$ A1

[3]

(c) The coordinates of A and B are $(6, 8, 7)$ and $(5, 8+\sqrt{3}, 7)$ respectively. (A1) for correct values

$\vec{CA}=(6\mathbf{i}+8\mathbf{j}+7\mathbf{k})-(4\mathbf{i}+8\mathbf{j}+7\mathbf{k})$

$\vec{CA}=2\mathbf{i}$ A1

$\vec{CB}=(5\mathbf{i}+(8+\sqrt{3})\mathbf{j}+7\mathbf{k})-(4\mathbf{i}+8\mathbf{j}+7\mathbf{k})$

$\vec{CB}=\mathbf{i}+\sqrt{3}\mathbf{j}$ A1

$\vec{CA}\cdot\vec{CB}=|\vec{CA}||\vec{CB}|\cos A\hat{C}B$ (M1) for valid approach

$2\mathbf{i}\cdot(\mathbf{i}+\sqrt{3}\mathbf{j})=|2\mathbf{i}||\mathbf{i}+\sqrt{3}\mathbf{j}|\cos A\hat{C}B$ (A1) for substitution

$(2)(1)+(0)(\sqrt{3})=(2)(\sqrt{1^2+(\sqrt{3})^2})\cos A\hat{C}B$

$2=4\cos A\hat{C}B$

$\cos A\hat{C}B=\frac{1}{2}$

$A\hat{C}B=60^\circ$ A1

[6]

(d) $CA = CB = 2$

The area of the triangle ABC

$= \frac{1}{2}(CA)(CB)\sin A\hat{C}B$ (M1) for valid approach

$= \frac{1}{2}(2)(2)\sin 60^\circ$ (A1) for substitution

$= 2\left(\frac{\sqrt{3}}{2}\right)$

$= \sqrt{3}$ A1

[3]

(e) Let h be the height of the prism ABCFED.

The triangle ABC is an equilateral triangle.

$\therefore 2\sqrt{3} + 3(2h) = 2(30 + \sqrt{3})$ M1A1

$2\sqrt{3} + 6h = 60 + 2\sqrt{3}$

$6h = 60$

$h = 10$ A1

The volume of the prism ABCFED

$= (\sqrt{3})(10)$

$= 10\sqrt{3}$ A1

[4]

3. (a) $L_1:\begin{cases} x=14-5t \\ y=18-6t \\ z=8-2t \end{cases}$ (M1) for valid approach

$(14-5t)+6=(8-2t)+6$ (M1) for setting equation

$20-5t=14-2t$

$6=3t$

$t=2$ A1

$\therefore \begin{cases} x=14-5(2)=4 \\ y=18-6(2)=6 \\ z=8-2(2)=4 \end{cases}$ (M1) for substitution

Thus, the coordinates of P are $(4, 6, 4)$. A1

[5]

(b) $a+6=3+6$ (M1) for setting equation

$a=3$ A1

[2]

(c) $\overrightarrow{\mathrm{RQ}}=-4\mathbf{i}-4\mathbf{j}-\mathbf{k}$

$\therefore \overrightarrow{\mathrm{OQ}}-\overrightarrow{\mathrm{OR}}=-4\mathbf{i}-4\mathbf{j}-\mathbf{k}$ (M1) for valid approach

$((14-5t)\mathbf{i}+(18-6t)\mathbf{j}+(8-2t)\mathbf{k})-(3\mathbf{i}+4\mathbf{j}+3\mathbf{k})$

$=-4\mathbf{i}-4\mathbf{j}-\mathbf{k}$

$14-5t-3=-4$

$-5t=-15$

$t=3$ (A1) for correct value

$\therefore \begin{cases} x=14-5(3)=-1 \\ y=18-6(3)=0 \\ z=8-2(3)=2 \end{cases}$ (M1) for substitution

Thus, the coordinates of Q are $(-1, 0, 2)$. A1

[4]

(d) $\overrightarrow{PQ} = (-\mathbf{i} + 2\mathbf{k}) - (4\mathbf{i} + 6\mathbf{j} + 4\mathbf{k})$

$\overrightarrow{PQ} = -5\mathbf{i} - 6\mathbf{j} - 2\mathbf{k}$ (A1) for correct values

$\overrightarrow{PR} = (3\mathbf{i} + 4\mathbf{j} + 3\mathbf{k}) - (4\mathbf{i} + 6\mathbf{j} + 4\mathbf{k})$

$\overrightarrow{PR} = -\mathbf{i} - 2\mathbf{j} - \mathbf{k}$ (A1) for correct values

$$\begin{cases} x + 6 = -4 \Rightarrow x = -10 \\ \dfrac{y+14}{2} = -4 \Rightarrow y = -22 \\ z + 6 = -4 \Rightarrow z = -10 \end{cases}$$

$\overrightarrow{PS} = (-10\mathbf{i} - 22\mathbf{j} - 10\mathbf{k}) - (4\mathbf{i} + 6\mathbf{j} + 4\mathbf{k})$

$\overrightarrow{PS} = -14\mathbf{i} - 28\mathbf{j} - 14\mathbf{k}$ (A1) for correct values

$$\begin{cases} x = 14 - 5(8) = -26 \\ y = 18 - 6(8) = -30 \\ z = 8 - 2(8) = -8 \end{cases}$$

$\overrightarrow{PT} = (-26\mathbf{i} - 30\mathbf{j} - 8\mathbf{k}) - (4\mathbf{i} + 6\mathbf{j} + 4\mathbf{k})$

$\overrightarrow{PT} = -30\mathbf{i} - 36\mathbf{j} - 12\mathbf{k}$ (A1) for correct values

The area of the quadrilateral QRST

= The area of PST − The area of PQR (M1) for valid approach

$= \frac{1}{2}\left|\overrightarrow{PS} \times \overrightarrow{PT}\right| - \frac{1}{2}\left|\overrightarrow{PQ} \times \overrightarrow{PR}\right|$

$= \frac{1}{2}\left|(-14\mathbf{i} - 28\mathbf{j} - 14\mathbf{k}) \times (-30\mathbf{i} - 36\mathbf{j} - 12\mathbf{k})\right|$

(A1) for substitution

$-\frac{1}{2}\left|(-5\mathbf{i} - 6\mathbf{j} - 2\mathbf{k}) \times (-\mathbf{i} - 2\mathbf{j} - \mathbf{k})\right|$

$= \frac{1}{2}(14)(6)\left|(-\mathbf{i} - 2\mathbf{j} - \mathbf{k}) \times (-5\mathbf{i} - 6\mathbf{j} - 2\mathbf{k})\right|$

$-\frac{1}{2}\left|(-5\mathbf{i} - 6\mathbf{j} - 2\mathbf{k}) \times (-\mathbf{i} - 2\mathbf{j} - \mathbf{k})\right|$

$= \frac{83}{2}\left|(-\mathbf{i} - 2\mathbf{j} - \mathbf{k}) \times (-5\mathbf{i} - 6\mathbf{j} - 2\mathbf{k})\right|$ (M1) for simplification

$$= \frac{83}{2}\left|\begin{pmatrix} (-2)(-2) - (-1)(-6) \\ (-1)(-5) - (-1)(-2) \\ (-1)(-6) - (-2)(-5) \end{pmatrix}\right|$$

$= \frac{83}{2}\left|-2\mathbf{i} + 3\mathbf{j} - 4\mathbf{k}\right|$

$= \frac{83}{2}\sqrt{(-2)^2+3^2+4^2}$

$= \frac{83\sqrt{29}}{2}$ A1

[8]

(e) $\frac{1}{3}\left(\frac{83\sqrt{29}}{2}\right)(\text{UQ}) = 162\sqrt{29}$ M1

$\frac{1}{6}\text{UQ} = 2$

$\text{UQ} = 12$

Thus, the shortest distance between U and QRST is 12. A1

[2]

4. (a) $\overrightarrow{BD} = \begin{pmatrix} 0 \\ -3 \\ 0 \end{pmatrix} - \begin{pmatrix} 3 \\ 0 \\ 3 \end{pmatrix}$

$\overrightarrow{BD} = \begin{pmatrix} -3 \\ -3 \\ -3 \end{pmatrix}$ A1

The vector equation of BD:

$\mathbf{r} = \begin{pmatrix} 3 \\ 0 \\ 3 \end{pmatrix} + t \begin{pmatrix} -3 \\ -3 \\ -3 \end{pmatrix}$

$\begin{cases} x = 3 - 3t \\ y = -3t \\ z = 3 - 3t \end{cases}$ A1

$\overrightarrow{AE} = \begin{pmatrix} 3-3t \\ -3t \\ 3-3t \end{pmatrix} - \begin{pmatrix} 3 \\ 0 \\ 0 \end{pmatrix}$

$\overrightarrow{AE} = \begin{pmatrix} -3t \\ -3t \\ 3-3t \end{pmatrix}$ A1

$\overrightarrow{AE} \cdot \overrightarrow{BD} = 0$

$\therefore (-3t)(-3) + (-3t)(-3) + (3-3t)(-3) = 0$ M1

$9t + 9t - 9 + 9t = 0$

$27t = 9$

$t = \frac{1}{3}$ A1

$\therefore \begin{cases} x = 3 - 3\left(\frac{1}{3}\right) = 2 \\ y = -3\left(\frac{1}{3}\right) = -1 \\ z = 3 - 3\left(\frac{1}{3}\right) = 2 \end{cases}$ M1

Therefore, the coordinates of E are $(2, -1, 2)$. AG

[6]

(b) $\overrightarrow{BA} = \begin{pmatrix} 3 \\ 0 \\ 0 \end{pmatrix} - \begin{pmatrix} 3 \\ 0 \\ 3 \end{pmatrix}$

$\overrightarrow{BA} = \begin{pmatrix} 0 \\ 0 \\ -3 \end{pmatrix}$ (A1) for correct values

$\overrightarrow{BC} = \begin{pmatrix} 0 \\ 0 \\ 3 \end{pmatrix} - \begin{pmatrix} 3 \\ 0 \\ 3 \end{pmatrix}$

$\overrightarrow{BC} = \begin{pmatrix} -3 \\ 0 \\ 0 \end{pmatrix}$ (A1) for correct values

$\mathbf{n}_1 = \overrightarrow{BA} \times \overrightarrow{BD}$ (M1) for valid approach

$\mathbf{n}_1 = -3\mathbf{k} \times (-3\mathbf{i} - 3\mathbf{j} - 3\mathbf{k})$

$\mathbf{n}_1 = \begin{pmatrix} (0)(-3) - (-3)(-3) \\ (-3)(-3) - (0)(-3) \\ (0)(-3) - (0)(-3) \end{pmatrix}$

$\mathbf{n}_1 = -9\mathbf{i} + 9\mathbf{j}$ A1

$\mathbf{n}_2 = \overrightarrow{BC} \times \overrightarrow{BD}$ (M1) for valid approach

$\mathbf{n}_2 = -3\mathbf{i} \times (-3\mathbf{i} - 3\mathbf{j} - 3\mathbf{k})$

$\mathbf{n}_2 = \begin{pmatrix} (0)(-3) - (0)(-3) \\ (0)(-3) - (-3)(-3) \\ (-3)(-3) - (0)(-3) \end{pmatrix}$

$\mathbf{n}_2 = -9\mathbf{j} + 9\mathbf{k}$ A1

$\mathbf{n}_1 \cdot \mathbf{n}_2 = |\mathbf{n}_1||\mathbf{n}_2| \cos\theta$ (M1) for valid approach

$(-9\mathbf{i} + 9\mathbf{j}) \cdot (-9\mathbf{j} + 9\mathbf{k}) = |-9\mathbf{i} + 9\mathbf{j}||-9\mathbf{j} + 9\mathbf{k}| \cos\theta$

$(-9)(0) + (9)(-9) + (0)(9)$

$= (\sqrt{(-9)^2 + 9^2})(\sqrt{(-9)^2 + 9^2}) \cos\theta$ A1

$-81 = 162 \cos\theta$

$\cos\theta = -\frac{1}{2}$

$\theta = 120^\circ$ A1

[9]

(c) The area of OABC

$= (OA)(OC)$

$= (3)(3)$

$= 9$ — (A1) for correct value

$\therefore \frac{1}{3}(9)(OF) = 15$ — (M1) for setting equation

$3OF = 15$

$OF = 5$ — A1

Thus, the possible coordinates of F are $(0, 5, 0)$ and $(0, -5, 0)$. — A1

The possible values of DF

$= 5 - (-3)$ or $= (-3) - (-5)$

$= 8$ or 2 — A2

[6]

Exercise 31

1. Let $\mathbf{n}_1$ and $\mathbf{n}_2$ be the normal vectors of $2x-9y+4z-1=0$ and $3x+14y+3z-3=0$ respectively.

$$\mathbf{n}_1=\begin{pmatrix}2\\-9\\4\end{pmatrix}$$

$$\mathbf{n}_2=\begin{pmatrix}3\\14\\3\end{pmatrix}$$ (A1) for correct values

Let θ be the acute angle between the planes.

$\mathbf{n}_1\cdot\mathbf{n}_2=|\mathbf{n}_1||\mathbf{n}_2|\cos\theta$ (M1) for valid approach

$(2)(3)+(-9)(14)+(4)(3)$

$=(\sqrt{2^2+(-9)^2+4^2})(\sqrt{3^2+14^2+3^2})\cos\theta$ (A1) for substitution

$-108=(\sqrt{101})(\sqrt{214})\cos\theta$

$\cos\theta=-\dfrac{108}{(\sqrt{101})(\sqrt{214})}$ A1

$\theta=137.2741785^\circ$

$\theta=137^\circ$ A1

[5]

2. By using row operations, the system $\left(\begin{array}{ccc|c} 5 & 3 & -2 & 7 \\ 4 & 2 & -4 & 5 \end{array}\right)$ is

reduced to $\left(\begin{array}{ccc|c} 1 & 0 & -4 & \frac{1}{2} \\ 0 & 1 & 6 & \frac{3}{2} \end{array}\right)$. (M1) for valid approach

$y+6z=\frac{3}{2}$

$y=\frac{3}{2}-6z$ A1

$x-4z=\frac{1}{2}$

$x=\frac{1}{2}+4z$ A1

Let $z=t$.

$x=\frac{1}{2}+4t$

$y=\frac{3}{2}-6t$

Thus, the vector equation of the line of intersection is

$\mathbf{r}=\begin{pmatrix} \frac{1}{2} \\ \frac{3}{2} \\ 0 \end{pmatrix}+t\begin{pmatrix} 4 \\ -6 \\ 1 \end{pmatrix}$. A2

[5]

3. By using row operations, the system $\left(\begin{array}{ccc|c} 4 & 2 & -1 & 5 \\ 3 & 1 & -2 & 2 \\ 1 & 1 & 1 & 3 \end{array}\right)$ is

reduced to $\left(\begin{array}{ccc|c} 1 & 0 & -\frac{3}{2} & -\frac{1}{2} \\ 0 & 1 & \frac{5}{2} & \frac{7}{2} \\ 0 & 0 & 0 & 0 \end{array}\right)$. (M1) for valid approach

$y+\frac{5}{2}z=\frac{7}{2}$

$y=\frac{7}{2}-\frac{5}{2}z$ A1

$x-\frac{3}{2}z=-\frac{1}{2}$

$x=-\frac{1}{2}+\frac{3}{2}z$ A1

Let $z=t$.

$x=-\frac{1}{2}+\frac{3}{2}t$, $y=\frac{7}{2}-\frac{5}{2}t$

Thus, the vector equation of the line of intersection is

$\mathbf{r}=\begin{pmatrix} -\frac{1}{2} \\ \frac{7}{2} \\ 0 \end{pmatrix}+t\begin{pmatrix} \frac{3}{2} \\ -\frac{5}{2} \\ 1 \end{pmatrix}$. A2

[5]

4. (a) By using row operations, the system

$\left(\begin{array}{ccc|c} 2 & -2 & -3 & 9 \\ 1 & -4 & -4 & 9 \\ 2 & 1 & 2 & -3 \end{array}\right)$ is reduced to $\left(\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -3 \end{array}\right)$. (M1) for valid approach

Thus, the coordinates of A are $(1, 1, -3)$. A3

[4]

(b) $\mathbf{r}=\begin{pmatrix} 1 \\ 1 \\ -3 \end{pmatrix}+t\begin{pmatrix} 2 \\ -2 \\ -3 \end{pmatrix}$ A2

[2]

Exercise 32

1.	(a)	$\mathbf{n} = (3\mathbf{i} - 2\mathbf{j}) \times (\mathbf{j} - 3\mathbf{k})$	(M1) for valid approach
		$\mathbf{n} = \begin{pmatrix} (-2)(-3) - (0)(1) \\ (0)(0) - (3)(-3) \\ (3)(1) - (-2)(0) \end{pmatrix}$	
		$\mathbf{n} = 6\mathbf{i} + 9\mathbf{j} + 3\mathbf{k}$	(A1) for correct values
		The Cartesian equation of the plane π:	
		$(x\mathbf{i} + y\mathbf{j} + z\mathbf{k}) \cdot (6\mathbf{i} + 9\mathbf{j} + 3\mathbf{k})$ $= (-6\mathbf{i} + 18\mathbf{k}) \cdot (6\mathbf{i} + 9\mathbf{j} + 3\mathbf{k})$	M1A1
		$6x + 9y + 3z = (-6)(6) + (0)(9) + (18)(3)$	
		$6x + 9y + 3z = 18$	
		$2x + 3y + z = 6$	A1
			[5]
	(b)	$\mathbf{r} = \begin{pmatrix} 1 \\ 3 \\ 13 \end{pmatrix} + t\begin{pmatrix} -10 \\ 5 \\ -13 \end{pmatrix}$	
		$\begin{cases} x = 1 - 10t \\ y = 3 + 5t \\ z = 13 - 13t \end{cases}$	
		$\therefore 2(1 - 10t) + 3(3 + 5t) + (13 - 13t) = 6$	(M1) for substitution
		$2 - 20t + 9 + 15t + 13 - 13t = 6$	
		$-18t = -18$	
		$t = 1$	(A1) for correct value
		$\therefore \begin{cases} x = 1 - 10(1) = -9 \\ y = 3 + 5(1) = 8 \\ z = 7 - 13(1) = -6 \end{cases}$	
		Thus, the coordinates of the point of intersection are $(-9, 8, -6)$.	A1
			[3]
	(c)	$a = 3$, $b = 2$, $c = 6$	A3
			[3]

(d) The volume of the pyramid OABC

$= \frac{1}{3}\left(\frac{(\text{OA})(\text{OB})}{2}\right)(\text{OC})$ (M1) for valid approach

$= \frac{1}{3}\left(\frac{(3)(2)}{2}\right)(6)$ (A1) for substitution

$= 6$ A1

[3]

(e) (i) $\overrightarrow{\text{CA}} = 3\mathbf{i} - 6\mathbf{k}$, $\overrightarrow{\text{CB}} = 2\mathbf{j} - 6\mathbf{k}$ A2

(ii) $\frac{1}{2}\left|\overrightarrow{\text{CA}} \times \overrightarrow{\text{CB}}\right| = \alpha\sqrt{14}$ (M1) for setting equation

$\frac{1}{2}\left|(3\mathbf{i} - 6\mathbf{k}) \times (2\mathbf{j} - 6\mathbf{k})\right| = \alpha\sqrt{14}$

$\frac{1}{2}\left|\begin{pmatrix} (0)(-6) - (-6)(2) \\ (-6)(0) - (3)(-6) \\ (3)(2) - (0)(0) \end{pmatrix}\right| = \alpha\sqrt{14}$

$\frac{1}{2}\left|12\mathbf{i} + 18\mathbf{j} + 6\mathbf{k}\right| = \alpha\sqrt{14}$ (A1) for correct values

$\frac{1}{2}(6\left|2\mathbf{i} + 3\mathbf{j} + \mathbf{k}\right|) = \alpha\sqrt{14}$

$3\sqrt{2^2 + 3^2 + 1^2} = \alpha\sqrt{14}$ M1

$3\sqrt{14} = \alpha\sqrt{14}$

$\therefore \alpha = 3$ A1

[6]

(f) Let h be the required perpendicular distance.

$\frac{1}{3}(3\sqrt{14})h = 6$ (M1) for setting equation

$h = \frac{6}{\sqrt{14}}$

$h = \frac{3\sqrt{14}}{11}$

Thus, the required perpendicular distance is $\frac{3\sqrt{14}}{11}$. A1

[2]

2. (a) $\mathbf{n} = (3\mathbf{i} + \mathbf{j}) \times (4\mathbf{i} + \mathbf{k})$ (M1) for valid approach

$$\mathbf{n} = \begin{pmatrix} (1)(1) - (0)(0) \\ (0)(4) - (3)(1) \\ (3)(0) - (1)(4) \end{pmatrix}$$

$\mathbf{n} = \mathbf{i} - 3\mathbf{j} - 4\mathbf{k}$ (A1) for correct values

The Cartesian equation of the plane π :

$(x\mathbf{i} + y\mathbf{j} + z\mathbf{k}) \cdot (\mathbf{i} - 3\mathbf{j} - 4\mathbf{k}) = (-6\mathbf{i} + 2\mathbf{j}) \cdot (\mathbf{i} - 3\mathbf{j} - 4\mathbf{k})$ M1A1

$x - 3y - 4z = (-6)(1) + (2)(-3) + (0)(-4)$

$x - 3y - 4z = -12$ A1

[5]

(b) $x - 3(0) - 4(0) = -12$

$x = -12$

$\therefore \text{OA} = 12$ (A1) for correct value

$0 - 3y - 4(0) = -12$

$y = 4$

$\therefore \text{OB} = 4$ (A1) for correct value

$0 - 3(0) - 4z = -12$

$z = 3$

$\therefore \text{OC} = 3$ (A1) for correct value

The volume of the pyramid OABC

$= \frac{1}{3}\left(\frac{(\text{OA})(\text{OB})}{2}\right)(\text{OC})$ (M1) for valid approach

$= \frac{1}{3}\left(\frac{(12)(4)}{2}\right)(3)$ A1

$= 24$ A1

[6]

(c) The vector equation of the line L :

$$\mathbf{r} = \begin{pmatrix} -12 \\ 0 \\ 0 \end{pmatrix} + t\begin{pmatrix} 1 \\ -3 \\ -4 \end{pmatrix}$$ A2

[2]

(d) (i) (12, 0, 0) A1

(ii) The vector equation of the line L':

$$\mathbf{r} = \begin{pmatrix} 12 \\ 0 \\ 0 \end{pmatrix} + s\begin{pmatrix} 1 \\ -3 \\ -4 \end{pmatrix}$$ (A1) for correct values

$$\begin{cases} x = 12 + s \\ y = -3s \\ z = -4s \end{cases}$$

$$x - 12 = -\frac{y}{3} = -\frac{z}{4}$$ A1

(iii) $$-\frac{\beta + 12}{3} = -\frac{\beta}{4}$$ (M1) for setting equation

$$4(\beta + 12) = 3\beta$$

$$4\beta + 48 = 3\beta$$

$$\beta = -48$$ A1

[5]

3. (a) By using row operations, the system

$$\left(\begin{array}{ccc|c} 3 & -1 & 2 & 12 \\ 3 & 1 & -2 & -12 \end{array}\right) \text{ is reduced to}$$

$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & -2 & -12 \end{array}\right).$$ M1

$y-2z=-12$

$y=-12+2z$ A1

$x=0$ A1

Let $z=t$.

$y=-12+2t$ A1

Thus, the vector equation of the line of intersection

is $\mathbf{r}=\begin{pmatrix} 0 \\ -12 \\ 0 \end{pmatrix}+t\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$. AG

[4]

(b) (i) $a=4$, $b=-12$, $c=6$, $\alpha=-4$ A4

(ii) Let O be the origin.

The volume of the pyramid $\text{A}'\text{ABC}$

$=\frac{1}{3}\left(\frac{(\text{A}'\text{A})(\text{OB})}{2}\right)(\text{OC})$ (M1) for valid approach

$=\frac{1}{3}\left(\frac{(4-(-4))(12)}{2}\right)(6)$ A1

$=96$ A1

[7]

(c) (i) $\overrightarrow{AC} = -4\mathbf{i} + 6\mathbf{k}$

$\overrightarrow{AC} \cdot (-\mathbf{i}) = \left|\overrightarrow{AC}\right| \left|-\mathbf{i}\right| \cos C\hat{A}A'$ (M1) for valid approach

$(-4\mathbf{i} + 6\mathbf{k}) \cdot (-\mathbf{i}) = (\sqrt{(-4)^2 + 6^2})(1) \cos C\hat{A}A'$ (A1) for substitution

$(-4)(-1) + (6)(0) = \sqrt{52} \cos C\hat{A}A'$

$\cos C\hat{A}A' = \dfrac{4}{\sqrt{52}}$

$C\hat{A}A' = 56.30993247^\circ$

$C\hat{A}A' = 56.3^\circ$ A1

(ii) $\because CA' = CA$

$\therefore C\hat{A}'A = 56.30993247^\circ$ (A1) for correct approach

$A\hat{C}A' + 56.30993247^\circ + 56.30993247^\circ = 180^\circ$

$A\hat{C}A' = 67.38013505^\circ$

$A\hat{C}A' = 67.4^\circ$ A1

[5]

(d) The vector equation of L:

$$\mathbf{r} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} + s \begin{pmatrix} 3 \\ -1 \\ 2 \end{pmatrix}$$

$\begin{cases} x = 3s \\ y = -s \\ z = 2s \end{cases}$ (A1) for correct approach

$\therefore 3(3s) + (-s) - 2(2s) = -12$ (A1) for substitution

$4s = -12$

$s = -3$

$\begin{cases} x = 3(-3) = -9 \\ y = -(-3) = 3 \\ z = 2(-3) = -6 \end{cases}$ M1

Thus, the coordinates of Q are $(-9, 3, -6)$. A1

[4]

4.	(a)	The coordinates of A, B and C′ are $(6, 0, 0)$, $(0, -4, 0)$ and $(0, 0, 3)$ respectively.	A1
		$\mathbf{n} = \overrightarrow{AB} \times \overrightarrow{AC'}$	M1
		$\mathbf{n} = (-6\mathbf{i} - 4\mathbf{j}) \times (-6\mathbf{i} + 3\mathbf{k})$	A1
		$\mathbf{n} = \begin{pmatrix} (-4)(3) - (0)(0) \\ (0)(-6) - (-6)(3) \\ (-6)(0) - (-4)(-6) \end{pmatrix}$	
		$\mathbf{n} = -12\mathbf{i} + 18\mathbf{j} - 24\mathbf{k}$	A1
		The Cartesian equation of the plane π_2:	
		$(x\mathbf{i} + y\mathbf{j} + z\mathbf{k}) \cdot (-12\mathbf{i} + 18\mathbf{j} - 24\mathbf{k})$ $= 6\mathbf{i} \cdot (-12\mathbf{i} + 18\mathbf{j} - 24\mathbf{k})$	M1A1
		$-12x + 18y - 24z = (6)(-12) + (0)(18) + (0)(-24)$	
		$-12x + 18y - 24z = -72$	
		$2x - 3y + 4z = 12$	AG
			[6]
	(b)	The coordinates of C are $(0, 0, -3)$.	(A1) for correct values
		The volume of the pyramid ABCC′	
		$= \frac{1}{3}\left(\frac{(CC')(OA)}{2}\right)(OB)$	(M1) for valid approach
		$= \frac{1}{3}\left(\frac{(3-(-3))(6)}{2}\right)(4)$	A1
		$= 24$	A1
			[4]
	(c)	$\mathbf{n}_1 = 2\mathbf{i} - 3\mathbf{j} - 4\mathbf{k}$	
		$\mathbf{n}_2 = 2\mathbf{i} - 3\mathbf{j} + 4\mathbf{k}$	(A1) for correct values
		Let θ be the obtuse angle between the planes.	
		$\mathbf{n}_1 \cdot \mathbf{n}_2 = \lvert\mathbf{n}_1\rvert\lvert\mathbf{n}_2\rvert\cos\theta$	(M1) for valid approach
		$(2)(2) + (-3)(-3) + (-4)(4)$ $= (\sqrt{2^2 + (-3)^2 + (-4)^2})(\sqrt{2^2 + (-3)^2 + 4^2})\cos\theta$	(A1) for substitution
		$-3 = (\sqrt{29})(\sqrt{29})\cos\theta$	
		$\cos\theta = -\frac{3}{29}$	A1
		$\theta = 95.93777245^\circ$	
		$\theta = 95.9^\circ$	A1
			[5]

(d) (i) $(0, -2, -1.5)$ A1

(ii) $\mathbf{n}_3 = \mathbf{n}_1 \times \mathbf{n}_2$ (M1) for valid approach

$\mathbf{n}_3 = (2\mathbf{i} - 3\mathbf{j} - 4\mathbf{k}) \times (2\mathbf{i} - 3\mathbf{j} + 4\mathbf{k})$

$$\mathbf{n}_3 = \begin{pmatrix} (-3)(4) - (-4)(-3) \\ (-4)(2) - (2)(4) \\ (2)(-3) - (-3)(2) \end{pmatrix}$$

$\mathbf{n}_3 = -24\mathbf{i} - 16\mathbf{j}$ A1

The vector equation of the line:

$$\mathbf{r} = \begin{pmatrix} 0 \\ -2 \\ -1.5 \end{pmatrix} + t \begin{pmatrix} -24 \\ -16 \\ 0 \end{pmatrix}$$ A1

$$\begin{cases} x = -24t \\ y = -2 - 16t \\ z = -1.5 \end{cases}$$

$\dfrac{x}{-24} = \dfrac{y+2}{-16}, \ z = -1.5$ A1

[5]

Chapter 9 Solution

Exercise 33

1. (a) $(\sqrt{3}+i)^3$

$= (\sqrt{3})^3 + \binom{3}{1}(\sqrt{3})^2 i + \binom{3}{2}(\sqrt{3})i^2 + i^3$ M1A1

$= 3\sqrt{3} + (3)(3)i + (3)(\sqrt{3})(-1) + (-i)$ M1A1

$= 8i$ A1

[5]

(b) -8 A1

[1]

2. $\frac{z}{1-z} = -1 - 0.5i$

$z = (-1-0.5i)(1-z)$ M1

$z = -1 + z - 0.5i + 0.5iz$

$1 + 0.5i = 0.5iz$

$z = \frac{1+0.5i}{0.5i}$ A1

$z = \frac{(1+0.5i)(-i)}{0.5i(-i)}$ M1

$z = \frac{-i-0.5i^2}{-0.5i^2}$

$z = \frac{-i+0.5}{0.5}$ A1

$z = 1 - 2i$

Thus, the imaginary part of z is -2. A1

[5]

3. Let $z = a + b\mathrm{i}$.

$z - |z| = -4(2 - \mathrm{i})$

$a + b\mathrm{i} - \sqrt{a^2 + b^2} = -8 + 4\mathrm{i}$ M1A1

$(a - \sqrt{a^2 + b^2}) + b\mathrm{i} = -8 + 4\mathrm{i}$

$b = 4$ A1

$a - \sqrt{a^2 + 4^2} = -8$

$a + 8 = \sqrt{a^2 + 16}$

$(a + 8)^2 = a^2 + 16$ M1

$a^2 + 16a + 64 = a^2 + 16$

$16a = -48$

$a = -3$

Thus, the real part of z is -3. A1

[5]

4. Let $z = a + b\mathrm{i}$.

$|z| - 18 = (z - 12)\mathrm{i}$

$\sqrt{a^2 + b^2} - 18 = (a + b\mathrm{i} - 12)\mathrm{i}$ M1A1

$\sqrt{a^2 + b^2} - 18 = a\mathrm{i} + b\mathrm{i}^2 - 12\mathrm{i}$

$\sqrt{a^2 + b^2} - 18 = -b + (a - 12)\mathrm{i}$ A1

$a - 12 = 0$

$a = 12$ A1

$\sqrt{12^2 + b^2} - 18 = -b$

$\sqrt{144 + b^2} = 18 - b$

$144 + b^2 = (18 - b)^2$ M1

$144 + b^2 = 324 - 36b + b^2$

$-180 = -36b$

$b = 5$

Thus, the imaginary part of z is 5. A1

[6]

Exercise 34

1. (a) $z_C = 3 - 3i + 2(12 + 5i)$ M1

$z_C = 27 + 7i$

$z_D = -2 + 9i + 2(12 + 5i)$ M1

$z_D = 22 + 19i$

Thus, the complex numbers represented by the points C and D are $27 + 7i$ and $22 + 19i$ respectively. A2

[4]

(b) The area of ABCD

$= (AB)(AD)$ M1

$= 2AB^2$

$= 2(\sqrt{5^2 + 12^2})^2$

$= 338$ A1

[2]

2. (a) $z_B = -18 + 10i + 20$ M1

$z_B = 2 + 10i$

$z_C = -18 + 10i + (10 - (20\sin 60^\circ)i)$ M1

$z_C = -8 + (10 - 10\sqrt{3})i$

Thus, the complex numbers represented by the points B and C are $2 + 10i$ and $-8 + (10 - 10\sqrt{3})i$ respectively. A2

[4]

(b) The area of ABC

$= \dfrac{(20)(20\sin 60^\circ)}{2}$ M1

$= \dfrac{(20)(10\sqrt{3})}{2}$

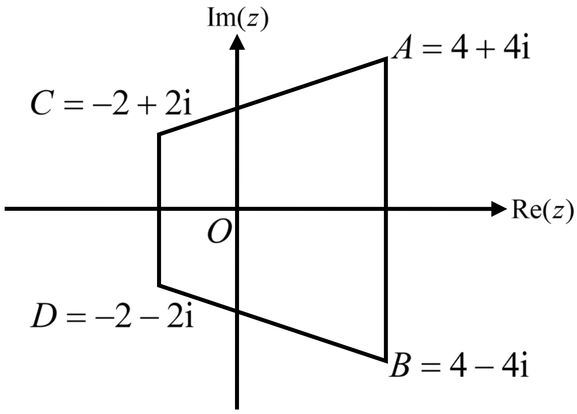
$= 100\sqrt{3}$ A1

[2]

3. (a) For any two correct points A1

For all correct points A1

For sketching a trapezium A1



[3]

(b) $\arg(\omega) = \arctan\left(\frac{2}{-2}\right)$ M1

$\arg(\omega) = \frac{3\pi}{4}$ A1

[2]

(c) The area of the quadrilateral ABDC

$= \frac{(4+8)(6)}{2}$ M1

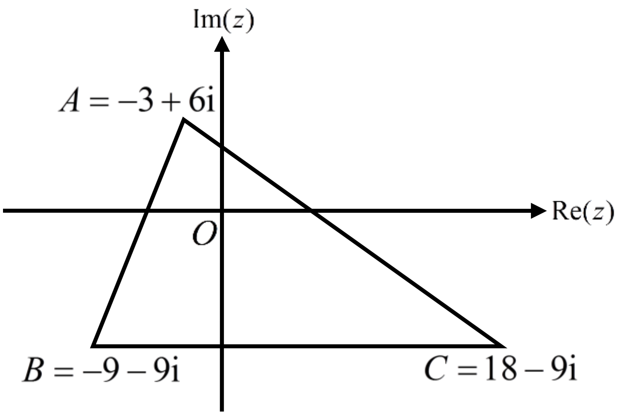
$= 36$ A1

[2]

4. (a) For any two correct points A1

For all correct points A1

For sketching a triangle A1



[3]

(b) $\arg(3z - 27i) = \arctan\left(\frac{-9}{-9}\right)$ M1

$\arg(\omega) = \frac{5\pi}{4}$ A1

[2]

(c) The area of the triangle ABC

$= \frac{(27)(15)}{2}$ M1

$= 202.5$ A1

[2]

Exercise 35

1. (a) $z_3 = \sqrt{3} + \mathrm{i}$ A1

[1]

(b) The cubic polynomial

$= (z - (-2))(z - (\sqrt{3} + \mathrm{i}))(z - (\sqrt{3} - \mathrm{i}))$ M1

$= (z + 2)(z^2 - 2\sqrt{3}z + 4)$

$= z^3 + (2 - 2\sqrt{3})z^2 + (4 - 4\sqrt{3})z + 8$

$= z^3 + 2(1 - \sqrt{3})z^2 + 4(1 - \sqrt{3})z + 8$

$\therefore b = 2$, $c = 4$ and $d = 8$ A3

[4]

(c) $z_3 = \sqrt{3} + \mathrm{i}$

$z_3 = (\sqrt{(\sqrt{3})^2 + 1^2})e^{\mathrm{i}\arctan\left(\frac{1}{\sqrt{3}}\right)}$

$z_3 = 2\mathrm{e}^{\frac{\pi}{6}\mathrm{i}}$ A2

[2]

2. (a) A polynomial function $f(x)$ of degree 4 has four roots. R1

There are two real roots only as the graph of $f(x)$ has only two x-intercepts. R1

Thus, there are two complex roots for the equation $f(x) = 0$. AG

[2]

(b) The another complex root is $3 + \mathrm{i}$. A1

$f(x) = (x - (-5))(x - 1)(x - (3 + \mathrm{i}))(x - (3 - \mathrm{i}))$ M1A1

$f(x) = (x + 5)(x - 1)(x^2 + 6x + 10)$ A1

[4]

3. The third root is $1-2\text{i}$. A1

The fourth root

$=3-4-(1+2\text{i})-(1-2\text{i})$

$=-3$ A1

The quartic polynomial

$=(z-4)(z-(-3))(z-(1+2\text{i}))(z-(1-2\text{i}))$ M1

$=(z^2-z-12)(z^2-2z+5)$ A1

$=z^4-3z^3-5z^2+19z-60$ A1

$\therefore b=-3$, $c=-5$, $d=19$ and $e=-60$

$b+c+d+e+49$

$=-3+(-5)+19+(-60)+49$ M1A1

$=0$ AG

[7]

4. The third root is $2+5\text{i}$. A1

The fourth root

$=\dfrac{174}{(-3)(2+5\text{i})(2-5\text{i})}$

$=-2$ A1

The quartic polynomial

$=(z-(-3))(z-(-2))(z-(2+5\text{i}))(z-(2-5\text{i}))$ M1

$=(z^2+5z+6)(z^2-4z+29)$ A1

$=z^4+z^3+15z^2+121z+174$ A1

$\therefore b=1$, $c=15$, $d=121$ and $e=174$

$\sqrt{b+c}+\sqrt{d}-15$

$=\sqrt{1+15}+\sqrt{121}-15$ M1A1

$=0$ AG

[7]

Exercise 36

1. (a) $z^9 = 1$

$z^9 = \cos 0 + \mathrm{i}\sin 0$ A1

$z = \cos\left(\dfrac{0+2k\pi}{9}\right) + \mathrm{i}\sin\left(\dfrac{0+2k\pi}{9}\right)$ M1

($k = 0, 1, 2, 3, 4, 5, 6, 7, 8$)

$z = \cos 0 + \mathrm{i}\sin 0$, $z = \cos\dfrac{2\pi}{9} + \mathrm{i}\sin\dfrac{2\pi}{9}$,

$z = \cos\dfrac{4\pi}{9} + \mathrm{i}\sin\dfrac{4\pi}{9}$, $z = \cos\dfrac{2\pi}{3} + \mathrm{i}\sin\dfrac{2\pi}{3}$,

$z = \cos\dfrac{8\pi}{9} + \mathrm{i}\sin\dfrac{8\pi}{9}$, $z = \cos\dfrac{10\pi}{9} + \mathrm{i}\sin\dfrac{10\pi}{9}$,

$z = \cos\dfrac{4\pi}{3} + \mathrm{i}\sin\dfrac{4\pi}{3}$, $z = \cos\dfrac{14\pi}{9} + \mathrm{i}\sin\dfrac{14\pi}{9}$

or $z = \cos\dfrac{16\pi}{9} + \mathrm{i}\sin\dfrac{16\pi}{9}$ A2

[4]

(b) (i) $(z^3 - 1)(z^6 + z^3 + 1)$

$= z^9 + z^6 + z^3 - z^6 - z^3 - 1$ M1

$= z^9 - 1$ A1

(ii) $z^6 + z^3 + 1 = 0$

$\dfrac{z^9 - 1}{z^3 - 1} = 0$, where $z \neq 1$, $z \neq \operatorname{cis}\dfrac{2\pi}{3}$ and

$z \neq \operatorname{cis}\dfrac{4\pi}{3}$ M1

$z = \operatorname{cis}\dfrac{2\pi}{9}$, $z = \operatorname{cis}\dfrac{4\pi}{9}$, $z = \operatorname{cis}\dfrac{8\pi}{9}$,

$z = \operatorname{cis}\dfrac{10\pi}{9}$, $z = \operatorname{cis}\dfrac{14\pi}{9}$ or $z = \operatorname{cis}\dfrac{16\pi}{9}$ A1

[4]

(c) (i) $\alpha^* = (w + w^4 + w^7)^*$

$\alpha^* = (w)^* + (w^4)^* + (w^7)^*$ M1

$\alpha^* = w^{9-1} + w^{9-4} + w^{9-7}$ A1

$\alpha^* = w^2 + w^5 + w^8$ A1

(ii) $1 + \alpha + \alpha^* = -\frac{b}{1}$ M1A1

$1 + w + w^4 + w^7 + w^2 + w^5 + w^8 = -b$

$1 + w(1 + w^3 + w^6) + w^2(1 + w^3 + w^6) = -b$ M1

$1 + 0 + 0 = -b$

$b = -1$ A1

(iii) $(1)(\alpha)(\alpha^*) = -\frac{d}{1}$ M1A1

$(w + w^4 + w^7)(w^2 + w^5 + w^8) = -d$

$(w)(1 + w^3 + w^6)(w^2)(1 + w^3 + w^6) = -d$ M1

$(w)(0)(w^2)(0) = -d$

$d = 0$ A1

(iv) $1^3 - 1^2 + c(1) + 0 = 0$ M1

$c = 0$ A1

[13]

2. (a) $z^5 + 1 = 0$

$z^5 = -1$

$z^5 = \cos\pi + i\sin\pi$ A1

$z = \cos\left(\frac{\pi + 2k\pi}{5}\right) + i\sin\left(\frac{\pi + 2k\pi}{5}\right)$ M1

($k = 0, 1, 2, 3, 4$)

$z = \cos\frac{\pi}{5} + i\sin\frac{\pi}{5}$, $z = \cos\frac{3\pi}{5} + i\sin\frac{3\pi}{5}$,

$z = \cos\pi + i\sin\pi$, $z = \cos\frac{7\pi}{5} + i\sin\frac{7\pi}{5}$ or

$z = \cos\frac{9\pi}{5} + i\sin\frac{9\pi}{5}$ A2

[4]

(b) (i) $(z+1)(z^4 - z^3 + z^2 - z + 1)$

$= z^5 - z^4 + z^3 - z^2 + z + z^4 - z^3 + z^2 - z + 1$ M1

$= z^5 + 1$ A1

(ii) $z^4 - z^3 + z^2 - z + 1 = 0$

$\frac{z^5 + 1}{z + 1} = 0$, where $z \neq -1$ M1

$z = \cos\frac{\pi}{5} + i\sin\frac{\pi}{5}$, $z = \cos\frac{3\pi}{5} + i\sin\frac{3\pi}{5}$,

$z = \cos\frac{7\pi}{5} + i\sin\frac{7\pi}{5}$ or

$z = \cos\frac{9\pi}{5} + i\sin\frac{9\pi}{5}$ A1

[4]

(c) (i) $\left(\cos\frac{\pi}{5}+\mathrm{i}\sin\frac{\pi}{5}\right)+\left(\cos\frac{3\pi}{5}+\mathrm{i}\sin\frac{3\pi}{5}\right)$

$+\left(\cos\frac{7\pi}{5}+\mathrm{i}\sin\frac{7\pi}{5}\right)+\left(\cos\frac{9\pi}{5}+\mathrm{i}\sin\frac{9\pi}{5}\right)$ M1A1

$=-\frac{-1}{1}$

$\cos\frac{\pi}{5}+\cos\frac{3\pi}{5}+\cos\frac{7\pi}{5}+\cos\frac{9\pi}{5}$

$+\mathrm{i}\left(\sin\frac{\pi}{5}+\sin\frac{3\pi}{5}+\sin\frac{7\pi}{5}+\sin\frac{9\pi}{5}\right)$

$=1$

$\therefore\cos\frac{\pi}{5}+\cos\frac{3\pi}{5}+\cos\frac{7\pi}{5}+\cos\frac{9\pi}{5}=1$ M1

$\cos\frac{\pi}{5}+\cos\frac{3\pi}{5}+\cos\left(2\pi-\frac{3\pi}{5}\right)$

$+\cos\left(2\pi-\frac{\pi}{5}\right)=1$

$2\cos\frac{\pi}{5}+2\cos\frac{3\pi}{5}=1$ M1

$\cos\frac{\pi}{5}+\cos\frac{3\pi}{5}=\frac{1}{2}$ A1

(ii) $\cos\frac{\pi}{5}+\cos\frac{3\pi}{5}=\frac{1}{2}$

$\sin\left(\frac{\pi}{2}-\frac{\pi}{5}\right)+\sin\left(\frac{\pi}{2}-\frac{3\pi}{5}\right)=\frac{1}{2}$ M1

$\sin\frac{3\pi}{10}+\sin\left(-\frac{\pi}{10}\right)=\frac{1}{2}$

$\sin\frac{3\pi}{10}-\sin\frac{\pi}{10}=\frac{1}{2}$ M1

$\therefore\sin\frac{\pi}{10}-\sin\frac{3\pi}{10}=-\frac{1}{2}$ A1

(iii) $\sum_{r=1}^{5}\cos\frac{(2r-1)\pi}{5}$

$=\cos\frac{\pi}{5}+\cos\frac{3\pi}{5}+\cos\pi+\cos\frac{7\pi}{5}+\cos\frac{9\pi}{5}$

$=\left(\cos\frac{\pi}{5}+\cos\frac{3\pi}{5}+\cos\frac{7\pi}{5}+\cos\frac{9\pi}{5}\right)$

$+\cos\pi$

$=1+(-1)$ M1

$=0$ A1

[10]

3. (a) (i) $z^3 - 1 = 0$

$z^3 = 1$

$z^3 = \cos 0 + \mathrm{i}\sin 0$ A1

$z = \cos\left(\frac{0+2k\pi}{3}\right) + \mathrm{i}\sin\left(\frac{0+2k\pi}{3}\right)$ M1

$(k = 0, 1, 2)$

$\therefore w = \cos\left(\frac{0+2(1)\pi}{3}\right) + \mathrm{i}\sin\left(\frac{0+2(1)\pi}{3}\right)$

$w = \cos\frac{2\pi}{3} + \mathrm{i}\sin\frac{2\pi}{3}$ A1

(ii) $w^3 - 1 = 0$

$(w-1)(w^2+w+1) = 0$ M1

$w^2 + w + 1 = 0$ A1

$\therefore w^3 + w^2 + w = 0$ AG

[5]

(b) (i) $\beta = \alpha^*$

$\beta = (w^3 + w^4)^*$ M1

$\beta = (1+w)^*$

$\beta = 1 + w^*$

$\beta = 1 + w^{3-1}$ A1

$\beta = 1 + w^2$ A1

(ii) $1 + (-1) + \alpha + \beta = -\frac{b}{1}$ M1A1

$1 + w + 1 + w^2 = -b$

$1 + 0 = -b$

$b = -1$ A1

(iii) $(1)(-1)(\alpha)(\beta) = \frac{e}{1}$ M1A1

$-(1+w)(1+w^2) = e$

$-(1 + w + w^2 + w^3) = e$

$-(0+1) = e$

$e = -1$ A1

(iv)	$(1)(-1)+(1)(\alpha)+(1)(\beta)$ $+(-1)(\alpha)+(-1)(\beta)+(\alpha)(\beta)=\frac{c}{1}$	M1
	$-1+\alpha\beta=c$	A1
	$-1+(1+w)(1+w^2)=c$	
	$-1+1+w+w^2+w^3=c$	A1
	$c=0$	AG
	$1^4-1^3+d(1)-1=0$	M1A1
	$d=1$	AG

[14]

4. (a) $z^7+1=0$

$z^7=-1$

$z^7=\cos\pi+\mathrm{i}\sin\pi$ A1

$z=\cos\left(\dfrac{\pi+2k\pi}{7}\right)+\mathrm{i}\sin\left(\dfrac{\pi+2k\pi}{7}\right)$ M1

($k=0, 1, 2, 3, 4, 5, 6$)

$z=\cos\dfrac{\pi}{7}+\mathrm{i}\sin\dfrac{\pi}{7}$, $z=\cos\dfrac{3\pi}{7}+\mathrm{i}\sin\dfrac{3\pi}{7}$,

$z=\cos\dfrac{5\pi}{7}+\mathrm{i}\sin\dfrac{5\pi}{7}$, $z=\cos\pi+\mathrm{i}\sin\pi$,

$z=\cos\dfrac{9\pi}{7}+\mathrm{i}\sin\dfrac{9\pi}{7}$, $z=\cos\dfrac{11\pi}{7}+\mathrm{i}\sin\dfrac{11\pi}{7}$ or

$z=\cos\dfrac{13\pi}{7}+\mathrm{i}\sin\dfrac{13\pi}{7}$ A2

[4]

(b) z^7+1

$=z^7-z^6+z^5-z^4+z^3-z^2+z$

$+z^6-z^5+z^4-z^3+z^2-z+1$ M1

$=z(z^6-z^5+z^4-z^3+z^2-z+1)$

$+z^6-z^5+z^4-z^3+z^2-z+1$

$=(z+1)(z^6-z^5+z^4-z^3+z^2-z+1)$ A1

$z^6-z^5+z^4-z^3+z^2-z+1=0$

$\dfrac{z^7+1}{z+1}=0$, where $z\neq-1$ M1

$z=\cos\dfrac{\pi}{7}+\mathrm{i}\sin\dfrac{\pi}{7}$, $z=\cos\dfrac{3\pi}{7}+\mathrm{i}\sin\dfrac{3\pi}{7}$,

$z=\cos\dfrac{5\pi}{7}+\mathrm{i}\sin\dfrac{5\pi}{7}$, $z=\cos\dfrac{9\pi}{7}+\mathrm{i}\sin\dfrac{9\pi}{7}$,

$z=\cos\dfrac{11\pi}{7}+\mathrm{i}\sin\dfrac{11\pi}{7}$ or $z=\cos\dfrac{13\pi}{7}+\mathrm{i}\sin\dfrac{13\pi}{7}$ A1

[4]

(c)	(i)	$(z-p)(z-q)=0$	M1
		$z^2-(p+q)z+pq=0$	
		$p+q=\lambda^4+\lambda^2-\lambda+\lambda^6-\lambda^5-\lambda^3$	M1
		$p+q=\lambda^6-\lambda^5+\lambda^4-\lambda^3+\lambda^2-\lambda$	
		$p+q=-1$	A1
		$pq=(\lambda^4+\lambda^2-\lambda)(\lambda^6-\lambda^5-\lambda^3)$	M1
		$pq=\lambda^{10}-\lambda^9-\lambda^7+\lambda^8-\lambda^7-\lambda^5$ $-\lambda^7+\lambda^6+\lambda^4$	M1
		$pq=\lambda^7(\lambda^3)-\lambda^7(\lambda^2)-\lambda^7$ $+\lambda^7(\lambda)-\lambda^7-\lambda^5-\lambda^7+\lambda^6+\lambda^4$	
		$pq=-\lambda^3+\lambda^2+1-\lambda+1-\lambda^5+1+\lambda^6+\lambda^4$	M1
		$pq=\lambda^6-\lambda^5+\lambda^4-\lambda^3+\lambda^2-\lambda+1+2$	
		$pq=0+2$	
		$pq=2$	A1
		$\therefore z^2+z+2=0$	A1
	(ii)	$(z-(p+1))(z-(q+1))=0$	M1
		$z^2-(p+1+q+1)z+(p+1)(q+1)=0$	
		$z^2-(p+q+2)z+(pq+p+q+1)=0$	A1
		$z^2-(-1+2)z+(2-1+1)=0$	M1
		$z^2-z+2=0$	A1

[12]

Exercise 37

1. (a) $z = \frac{2+i}{2-i}$

$z = \frac{(2+i)(2+i)}{(2-i)(2+i)}$ M1

$z = \frac{4+4i+i^2}{5}$

$z = \frac{4+4i+(-1)}{5}$

$z = \frac{3}{5} + \frac{4}{5}i$ A1

[2]

(b) The modulus of z

$= \sqrt{\left(\frac{3}{5}\right)^2 + \left(\frac{4}{5}\right)^2}$ M1

$= \sqrt{1}$

$= 1$ A1

[2]

(c) The argument of z

$= \arctan\left(\frac{\frac{4}{5}}{\frac{3}{5}}\right)$ M1

$= \arctan\left(\frac{4}{3}\right)$

$= 0.927295218$

$= 0.927 \text{ rad}$ A1

[2]

2. $\omega^* = (1+\cos\theta) + \mathrm{i}\sin\theta$ A1

$(\omega^*)^2 = ((1+\cos\theta) + \mathrm{i}\sin\theta)^2$ M1

$(\omega^*)^2 = (1+\cos\theta)^2 + 2\mathrm{i}\sin\theta(1+\cos\theta) + \mathrm{i}^2\sin^2\theta$

$(\omega^*)^2 = 1 + 2\cos\theta + \cos^2\theta + 2\mathrm{i}\sin\theta(1+\cos\theta) - \sin^2\theta$

$(\omega^*)^2 = 2\cos\theta + 2\cos^2\theta + 2\mathrm{i}\sin\theta(1+\cos\theta)$ (A1) for simplification

$(\omega^*)^2 = 2\cos\theta(1+\cos\theta) + 2\mathrm{i}\sin\theta(1+\cos\theta)$ A1

The modulus of $(\omega^*)^2$

$= \sqrt{(2\cos\theta(1+\cos\theta))^2 + (2\sin\theta(1+\cos\theta))^2}$ M1

$= \sqrt{4\cos^2\theta(1+\cos\theta)^2 + 4\sin^2\theta(1+\cos\theta)^2}$

$= \sqrt{4(1+\cos\theta)^2}$

$= 2|1+\cos\theta|$ A1

The argument of $(\omega^*)^2$

$= \arctan\left(\dfrac{2\sin\theta(1+\cos\theta)}{2\cos\theta(1+\cos\theta)}\right)$ M1

$= \arctan(\tan\theta)$

$= \theta$ A1

[8]

3. $1+\omega=(1+\sin\theta)+\mathrm{i}\cos\theta$

$(1+\omega)^2=((1+\sin\theta)+\mathrm{i}\cos\theta)^2$ M1A1

$(1+\omega)^2=(1+\sin\theta)^2+2\mathrm{i}\cos\theta(1+\sin\theta)+\mathrm{i}^2\cos^2\theta$

$(1+\omega)^2=1+2\sin\theta+\sin^2\theta+2\mathrm{i}\cos\theta(1+\sin\theta)-\cos^2\theta$

$(1+\omega)^2=2\sin\theta+2\sin^2\theta+2\mathrm{i}\cos\theta(1+\sin\theta)$ (A1) for simplification

$(1+\omega)^2=2\sin\theta(1+\sin\theta)+2\mathrm{i}\cos\theta(1+\sin\theta)$ A1

The modulus of $(1+\omega)^2$

$=\sqrt{(2\sin\theta(1+\sin\theta))^2+(2\cos\theta(1+\sin\theta))^2}$ M1

$=\sqrt{4\sin^2\theta(1+\sin\theta)^2+4\cos^2\theta(1+\sin\theta)^2}$

$=\sqrt{4(1+\sin\theta)^2}$

$=2|1+\sin\theta|$ A1

The argument of $(1+\omega)^2$

$=\arctan\left(\dfrac{2\cos\theta(1+\sin\theta)}{2\sin\theta(1+\sin\theta)}\right)$ M1

$=\arctan(\cot\theta)$

$=\arctan\left(\tan\left(\dfrac{\pi}{2}-\theta\right)\right)$

$=\dfrac{\pi}{2}-\theta$ A1

$\therefore (1+\omega)^2=2|1+\sin\theta|e^{\left(\frac{\pi}{2}-\theta\right)\mathrm{i}}$ A1

[9]

4. (a)

$z_1 + iz_2 = 0$

$z_1 = -iz_2$

$(-iz_2)z_2 = -2\sqrt{a} + 2i$ M1

$-iz_2{}^2 = -2\sqrt{a} + 2i$

$z_2{}^2 = \dfrac{2\sqrt{a}}{i} - 2$ A1

$z_2{}^2 = -2 - 2\sqrt{a}i$

$\left|z_2{}^2\right| = \left|-2 - 2\sqrt{a}i\right|$

$\left|z_2\right|^2 = \sqrt{(-2)^2 + (-2\sqrt{a})^2}$ M1

$2^2 = \sqrt{4 + 4a}$

$16 = 4 + 4a$

$12 = 4a$

$a = 3$ A1

[4]

(b)

$\left|z_1\right| = \left|-iz_2\right|$

$\left|z_1\right| = \left|-i\right|\left|z_2\right|$ M1

$\left|z_1\right| = (1)(2)$

$\left|z_1\right| = 2$ A1

[2]

(c)

$z_2{}^2 = -2 - 2\sqrt{3}i$

$\arg(z_2{}^2) = \arctan\left(\dfrac{-2\sqrt{3}}{-2}\right)$ M1

$2\arg(z_2) = \arctan(\sqrt{3})$

$2\arg(z_2) = \dfrac{\pi}{3}$

$\arg(z_2) = \dfrac{\pi}{6}$ A1

$\therefore z_2 = 2e^{\frac{\pi}{6}i}$ A1

[3]

Exercise 38

1. (a) $(-\text{i})^4 - 8(-\text{i})^3 + 13(-\text{i})^2 - 8(-\text{i}) + 12$ M1

$= 1 - 8\text{i} + 13(-1) - 8(-\text{i}) + 12$ A1

$= 1 - 8\text{i} - 13 + 8\text{i} + 12$

$= 0$

Thus, $-\text{i}$ is a root of the equation. AG

[2]

(b) The product of roots

$= \frac{12}{1}$

$= 12$ A1

[1]

(c) The second root is i. A1

Let $x^4 - 8x^3 + 13x^2 - 8x + 12$

$= (x^2 + bx + c)(x - \text{i})(x - (-\text{i}))$.

$x^4 - 8x^3 + 13x^2 - 8x + 12 = (x^2 + bx + c)(x^2 + 1)$ M1

$x^4 - 8x^3 + 13x^2 - 8x + 12$

$= x^4 + bx^3 + (c+1)x^2 + bx + c$ A1

$\therefore b = -8$ and $c = 12$

$x^2 - 8x + 12 = (x-2)(x-6)$

Thus, the other two roots are 2 and 6. A2

[5]

2. The third root is $a - b\text{i}$. A1

$4 + (a + b\text{i}) + (a - b\text{i}) = -\frac{-12}{1}$ M1A1

$4 + 2a = 12$

$2a = 8$

$a = 4$ A1

$(4)(4 + b\text{i})(4 - b\text{i}) = -\frac{-68}{1}$ A1

$(4)(16 + b^2) = 68$

$16 + b^2 = 17$

$b^2 = 1$

$b = -1$ (*Rejected*) or $b = 1$ A1

[6]

3. The other two roots are $a-b\text{i}$ and $b-a\text{i}$. A1

$(a+b\text{i})+(a-b\text{i})+(b+a\text{i})+(b-a\text{i})=-\frac{-14}{1}$ M1A1

$2a+2b=14$

$b=7-a$

$(a+b\text{i})(a-b\text{i})(b+a\text{i})(b-a\text{i})=\frac{841}{1}$ A1

$(a^2+b^2)(b^2+a^2)=841$

$(a^2+b^2)^2=841$

$a^2+b^2=29$

$\therefore a^2+(7-a)^2=29$ M1

$a^2+49-14a+a^2=29$

$2a^2-14a+20=0$ A1

$2(a-2)(a-5)=0$

$a=2$ or $a=5$

$b=5$ (*Rejected*) or $b=2$

Thus, $a=5$ and $b=2$. A2

[8]

4. The second root is $a-b\text{i}$. A1

The third and the fourth root are 6 as the graph of $y=P(x)$ touches the x-axis at $(6, 0)$. A1

$(a+b\text{i})+(a-b\text{i})+6+6=-\frac{-16}{1}$ M1A1

$2a+12=16$

$2a=4$

$a=2$ A1

$(2+b\text{i})(2-b\text{i})(6)(6)=\frac{468}{1}$ A1

$(4+b^2)(36)=468$

$4+b^2=13$

$b^2=9$

$b=-3$ (*Rejected*) or $b=3$ A1

[7]

Exercise 39

1. (a) (i) $(\cos\theta + i\sin\theta)^4$

$= \cos^4\theta + \binom{4}{1} i\cos^3\theta\sin\theta$

$+ \binom{4}{2} i^2\cos^2\theta\sin^2\theta + \binom{4}{3} i^3\cos\theta\sin^3\theta$ A1

$+ i^4\sin^4\theta$

$= \cos^4\theta + 4i\cos^3\theta\sin\theta - 6\cos^2\theta\sin^2\theta$

$-4i\cos\theta\sin^3\theta + \sin^4\theta$ A1

(ii) $(\cos\theta + i\sin\theta)^4 = \cos 4\theta + i\sin 4\theta$ A1

$\cos^4\theta + 4i\cos^3\theta\sin\theta - 6\cos^2\theta\sin^2\theta$

$-4i\cos\theta\sin^3\theta + \sin^4\theta = \cos 4\theta + i\sin 4\theta$ M1

$\cos^4\theta - 6\cos^2\theta\sin^2\theta + \sin^4\theta$

$+ i(4\cos^3\theta\sin\theta - 4\cos\theta\sin^3\theta)$

$= \cos 4\theta + i\sin 4\theta$

$\therefore \cos 4\theta = \cos^4\theta - 6\cos^2\theta\sin^2\theta + \sin^4\theta$

and $\sin 4\theta = 4\cos^3\theta\sin\theta - 4\cos\theta\sin^3\theta$ A2

[6]

(b) $z^4 + 16\text{i} = 0$

$z^4 = -16\text{i}$

$z^4 = 2^4\left(\cos\frac{3\pi}{2} + \text{i}\sin\frac{3\pi}{2}\right)$ A1

$z = 2\left(\cos\left(\frac{\frac{3\pi}{2} + 2k\pi}{4}\right) + \text{i}\sin\left(\frac{\frac{3\pi}{2} + 2k\pi}{4}\right)\right)$ M1

$(k = 0, 1, 2, 3)$

$\therefore z = 2\left(\cos\left(\frac{\frac{3\pi}{2} + 2(0)\pi}{4}\right) + \text{i}\sin\left(\frac{\frac{3\pi}{2} + 2(0)\pi}{4}\right)\right)$

$z = 2\left(\cos\frac{3\pi}{8} + \text{i}\sin\frac{3\pi}{8}\right)$

$\therefore r = 2,\ \alpha = \frac{3\pi}{8}$ A2

[4]

(c) $\cos 4\left(\frac{3\pi}{8}\right) = \cos^4\frac{3\pi}{8} - 6\cos^2\frac{3\pi}{8}\sin^2\frac{3\pi}{8} + \sin^4\frac{3\pi}{8}$ M1

$0 = \cos^4\frac{3\pi}{8} - 6\cos^2\frac{3\pi}{8}\left(1-\cos^2\frac{3\pi}{8}\right)$

$+\left(1-\cos^2\frac{3\pi}{8}\right)^2$ A1

$0 = \cos^4\frac{3\pi}{8} - 6\cos^2\frac{3\pi}{8} + 6\cos^4\frac{3\pi}{8} + 1$

$-2\cos^2\frac{3\pi}{8} + \cos^4\frac{3\pi}{8}$

$0 = 8\cos^4\frac{3\pi}{8} - 8\cos^2\frac{3\pi}{8} + 1$ A1

$\cos^2\frac{3\pi}{8} = \frac{-(-8)\pm\sqrt{(-8)^2-4(8)(1)}}{2(8)}$ M1

$\cos^2\frac{3\pi}{8} = \frac{8\pm\sqrt{32}}{16}$

As $\frac{\pi}{3} < \frac{3\pi}{8} < \frac{\pi}{2}$, $0 < \cos\frac{3\pi}{8} < \frac{1}{2}$ and

$0 < \cos^2\frac{3\pi}{8} < \frac{1}{4}$.

$\therefore \cos^2\frac{3\pi}{8} = \frac{8+\sqrt{32}}{16}$ (*Rejected*) or

$\cos^2\frac{3\pi}{8} = \frac{8-\sqrt{32}}{16}$. A1

$\cos\frac{3\pi}{8} = \sqrt{\frac{8-4\sqrt{2}}{16}}$

$\cos\frac{3\pi}{8} = \sqrt{\frac{2-\sqrt{2}}{4}}$ A1

[6]

(d) $\sin\frac{3\pi}{8}\cos\frac{3\pi}{8}\left(\cos\frac{3\pi}{8}+\sin\frac{3\pi}{8}\right)\left(\cos\frac{3\pi}{8}-\sin\frac{3\pi}{8}\right)$

$=\sin\frac{3\pi}{8}\cos\frac{3\pi}{8}\left(\cos^2\frac{3\pi}{8}-\sin^2\frac{3\pi}{8}\right)$ M1

$=\cos^3\frac{3\pi}{8}\sin\frac{3\pi}{8}-\cos\frac{3\pi}{8}\sin^3\frac{3\pi}{8}$

$=\dfrac{4\cos^3\frac{3\pi}{8}\sin\frac{3\pi}{8}-4\cos\frac{3\pi}{8}\sin^3\frac{3\pi}{8}}{4}$ A1

$=\dfrac{\sin 4\left(\frac{3\pi}{8}\right)}{4}$ A1

$=-\frac{1}{4}$ AG

[3]

2. (a) $(\cos\theta + \mathrm{i}\sin\theta)^8$

$= \cos^8\theta + \binom{8}{1}\mathrm{i}\cos^7\theta\sin\theta + \binom{8}{2}\mathrm{i}^2\cos^6\theta\sin^2\theta$

$+\binom{8}{3}\mathrm{i}^3\cos^5\theta\sin^3\theta + \binom{8}{4}\mathrm{i}^4\cos^4\theta\sin^4\theta$

$+\binom{8}{5}\mathrm{i}^5\cos^3\theta\sin^5\theta + \binom{8}{6}\mathrm{i}^6\cos^2\theta\sin^6\theta$ A2

$+\binom{8}{7}\mathrm{i}^7\cos\theta\sin^7\theta + \mathrm{i}^8\sin^8\theta$

$= \cos^8\theta + 8\mathrm{i}\cos^7\theta\sin\theta - 28\cos^6\theta\sin^2\theta$

$-56\mathrm{i}\cos^5\theta\sin^3\theta + 70\cos^4\theta\sin^4\theta$ A1

$+56\mathrm{i}\cos^3\theta\sin^5\theta - 28\cos^2\theta\sin^6\theta$

$-8\mathrm{i}\cos\theta\sin^7\theta + \sin^8\theta$

$\therefore \cos 8\theta + \mathrm{i}\sin 8\theta = \cos^8\theta - 28\cos^6\theta\sin^2\theta$

$+70\cos^4\theta\sin^4\theta - 28\cos^2\theta\sin^6\theta + \sin^8\theta$ M1

$+\mathrm{i}(8\cos^7\theta\sin\theta - 56\cos^5\theta\sin^3\theta$

$+56\cos^3\theta\sin^5\theta - 8\cos\theta\sin^7\theta)$

$\sin 8\theta = 8\cos^7\theta\sin\theta - 56\cos^5\theta\sin^3\theta$ A1

$+56\cos^3\theta\sin^5\theta - 8\cos\theta\sin^7\theta$

[5]

(b) $z^8 + 6561 = 0$

$z^8 = -6561$

$z^8 = 3^8(\cos\pi + \mathrm{i}\sin\pi)$ A1

$z = 3\left(\cos\left(\frac{\pi + 2k\pi}{8}\right) + \mathrm{i}\sin\left(\frac{\pi + 2k\pi}{8}\right)\right)$ M1

$(k = 0, 1, 2, 3, 4, 5, 6, 7)$

$\therefore z_1 = 3\left(\cos\left(\frac{\pi + 2(7)\pi}{8}\right) + \mathrm{i}\sin\left(\frac{\pi + 2(7)\pi}{8}\right)\right)$

$z_1 = 3\left(\cos\frac{15\pi}{8} + \mathrm{i}\sin\frac{15\pi}{8}\right)$

$z_1 = 3\left(\cos\left(-\frac{\pi}{8}\right) + \mathrm{i}\sin\left(-\frac{\pi}{8}\right)\right)$

$\therefore r = 3,\ \alpha_1 = -\frac{\pi}{8}$ A2

[4]

$$\sin 8\left(-\frac{\pi}{8}\right)=8\cos^7\left(-\frac{\pi}{8}\right)\sin\left(-\frac{\pi}{8}\right)$$

(c) $$-56\cos^5\left(-\frac{\pi}{8}\right)\sin^3\left(-\frac{\pi}{8}\right)$$ M1

$$+56\cos^3\left(-\frac{\pi}{8}\right)\sin^5\left(-\frac{\pi}{8}\right)-8\cos\left(-\frac{\pi}{8}\right)\sin^7\left(-\frac{\pi}{8}\right)$$

$$0=8\cos^7\alpha_1\sin\alpha_1-56\cos^5\alpha_1\sin^3\alpha_1$$
$$+56\cos^3\alpha_1\sin^5\alpha_1-8\cos\alpha_1\sin^7\alpha_1$$ A1

$$0=\cos^6\alpha_1-7\cos^4\alpha_1\sin^2\alpha_1$$
$$+7\cos^2\alpha_1\sin^4\alpha_1-\sin^6\alpha_1$$ A1

$$1-7\tan^2\alpha_1+7\tan^4\alpha_1-\tan^6\alpha_1=0$$ AG

[3]

(d) (i) 3 A1

(ii) Rectangle A1

(iii) Trapezium A1

(iv) $$z_2=3\left(\cos\left(\frac{\pi+2(6)\pi}{8}\right)+\mathrm{i}\sin\left(\frac{\pi+2(6)\pi}{8}\right)\right)$$ M1

$$z_2=3\left(\cos\frac{13\pi}{8}+\mathrm{i}\sin\frac{13\pi}{8}\right)$$

$$z_2=3\left(\cos\left(-\frac{3\pi}{8}\right)+\mathrm{i}\sin\left(-\frac{3\pi}{8}\right)\right)$$

$$\arg(z_2)=-\frac{3\pi}{8}$$

$$\therefore \arg(z_2^*)=\frac{3\pi}{8}$$

$$\mathrm{B\hat{O}E}=\frac{3\pi}{8}-\left(-\frac{3\pi}{8}\right)=\frac{3\pi}{4}$$ A1

$$\mathrm{BE}^2=3^2+3^2-2(3)(3)\cos\frac{3\pi}{4}$$ A1

$$\mathrm{BE}=\sqrt{9+9-18\cos\frac{3\pi}{4}}$$

$$\mathrm{BE}=5.543277195$$

$$\mathrm{BE}=5.54$$ A1

[7]

3. (a) When $n=1$,

$\text{L.H.S.}=(-\cos\theta+\mathrm{i}\sin\theta)^2$

$\text{L.H.S.}=\cos^2\theta-2\mathrm{i}\sin\theta\cos\theta+\mathrm{i}^2\sin^2\theta$

$\text{L.H.S.}=\cos^2\theta-\sin^2\theta-2\mathrm{i}\sin\theta\cos\theta$

$\text{L.H.S.}=\cos 2\theta-\mathrm{i}\sin 2\theta$

$\text{R.H.S.}=\cos 2\theta-\mathrm{i}\sin 2\theta$

Thus, the statement is true when $n=1$. R1

Assume that the statement is true when $n=k$. M1

$(-\cos\theta+\mathrm{i}\sin\theta)^{2k}=\cos 2k\theta-\mathrm{i}\sin 2k\theta$

When $n=k+1$,

$(-\cos\theta+\mathrm{i}\sin\theta)^{2(k+1)}$

$=(-\cos\theta+\mathrm{i}\sin\theta)^{2k}(-\cos\theta+\mathrm{i}\sin\theta)^2$ M1

$(-\cos\theta+\mathrm{i}\sin\theta)^{2(k+1)}$

$=(\cos 2k\theta-\mathrm{i}\sin 2k\theta)(\cos 2\theta-\mathrm{i}\sin 2\theta)$ A1

$(-\cos\theta+\mathrm{i}\sin\theta)^{2(k+1)}=\cos 2k\theta\cos 2\theta$

$-\mathrm{i}\cos 2k\theta\sin 2\theta-\mathrm{i}\sin 2k\theta\cos 2\theta+\mathrm{i}^2\sin 2k\theta\sin 2\theta$

$(-\cos\theta+\mathrm{i}\sin\theta)^{2(k+1)}$

$=\cos 2k\theta\cos 2\theta-\sin 2k\theta\sin 2\theta$ A1

$-\mathrm{i}(\cos 2k\theta\sin 2\theta+\sin 2k\theta\cos 2\theta)$

$(-\cos\theta+\mathrm{i}\sin\theta)^{2(k+1)}$

$=\cos(2k\theta+2\theta)-\mathrm{i}\sin(2k\theta+2\theta)$

$(-\cos\theta+\mathrm{i}\sin\theta)^{2(k+1)}$

$=\cos 2(k+1)\theta-\mathrm{i}\sin 2(k+1)\theta$

Thus, the statement is true when $n=k+1$.

Therefore, the statement is true for all $n\in\mathbb{Z}^+$. R1

[6]

(b) $(-\cos\theta + \mathrm{i}\sin\theta)^8$

$= (-\cos\theta)^8 + \binom{8}{1}\mathrm{i}(-\cos\theta)^7\sin\theta$

$+\binom{8}{2}\mathrm{i}^2(-\cos\theta)^6\sin^2\theta + \binom{8}{3}\mathrm{i}^3(-\cos\theta)^5\sin^3\theta$

$+\binom{8}{4}\mathrm{i}^4(-\cos\theta)^4\sin^4\theta + \binom{8}{5}\mathrm{i}^5(-\cos\theta)^3\sin^5\theta$ A2

$+\binom{8}{6}\mathrm{i}^6(-\cos\theta)^2\sin^6\theta + \binom{8}{7}\mathrm{i}^7(-\cos\theta)\sin^7\theta$

$+\mathrm{i}^8\sin^8\theta$

$= \cos^8\theta - 8\mathrm{i}\cos^7\theta\sin\theta - 28\cos^6\theta\sin^2\theta$

$+56\mathrm{i}\cos^5\theta\sin^3\theta + 70\cos^4\theta\sin^4\theta$ A1

$-56\mathrm{i}\cos^3\theta\sin^5\theta - 28\cos^2\theta\sin^6\theta$

$+8\mathrm{i}\cos\theta\sin^7\theta + \sin^8\theta$

$\therefore \cos 8\theta - \mathrm{i}\sin 8\theta = \cos^8\theta - 28\cos^6\theta\sin^2\theta$

$+70\cos^4\theta\sin^4\theta - 28\cos^2\theta\sin^6\theta + \sin^8\theta$ M1

$+\mathrm{i}(-8\cos^7\theta\sin\theta + 56\cos^5\theta\sin^3\theta$

$-56\cos^3\theta\sin^5\theta + 8\cos\theta\sin^7\theta)$

$\cos 8\theta = \cos^8\theta - 28\cos^6\theta\sin^2\theta$ A1

$+70\cos^4\theta\sin^4\theta - 28\cos^2\theta\sin^6\theta + \sin^8\theta$

[5]

(c) $\cos 8\theta = 41\cos^4\theta\sin^4\theta - 28\cos^2\theta\sin^6\theta + \sin^8\theta$

$\therefore \cos^8\theta - 28\cos^6\theta\sin^2\theta$

$+70\cos^4\theta\sin^4\theta - 28\cos^2\theta\sin^6\theta + \sin^8\theta$ M1

$= 41\cos^4\theta\sin^4\theta - 28\cos^2\theta\sin^6\theta + \sin^8\theta$

$\cos^8\theta - 28\cos^6\theta\sin^2\theta + 29\cos^4\theta\sin^4\theta = 0$

$\cos^4\theta - 28\cos^2\theta\sin^2\theta + 29\sin^4\theta = 0$ A1

$(\cos^2\theta)^2 - 28\cos^2\theta\sin^2\theta + 29(\sin^2\theta)^2 = 0$

$(\cos^2\theta - \sin^2\theta)(\cos^2\theta - 28\sin^2\theta) = 0$ A1

$\cos^2\theta - \sin^2\theta = 0$ or $\cos^2\theta - 28\sin^2\theta = 0$

$\tan^2\theta = 1$ or $\tan^2\theta = \frac{1}{28}$ A1

$\tan\theta = -1$ (*Rejected*), $\tan\theta = -\frac{1}{\sqrt{28}}$ (*Rejected*),

$\tan\theta = \frac{1}{\sqrt{28}}$ or $\tan\theta = 1$

$\theta = 0.186779461$ or $\theta = \frac{\pi}{4}$

$\theta = 0.187$ or $\theta = \frac{\pi}{4}$ A2

[6]

4. (a) $\left(\cos\frac{\theta}{5}+\mathrm{i}\sin\frac{\theta}{5}\right)^5$

$= \cos^5\frac{\theta}{5}+\binom{5}{1}\mathrm{i}\cos^4\frac{\theta}{5}\sin\frac{\theta}{5}+\binom{5}{2}\mathrm{i}^2\cos^3\frac{\theta}{5}\sin^2\frac{\theta}{5}$

$+\binom{5}{3}\mathrm{i}^3\cos^2\frac{\theta}{5}\sin^3\frac{\theta}{5}+\binom{5}{4}\mathrm{i}^4\cos\frac{\theta}{5}\sin^4\frac{\theta}{5}$ A2

$+\mathrm{i}^5\sin^5\frac{\theta}{5}$

$= \cos^5\frac{\theta}{5}+5\mathrm{i}\cos^4\frac{\theta}{5}\sin\frac{\theta}{5}-10\cos^3\frac{\theta}{5}\sin^2\frac{\theta}{5}$

$-10\mathrm{i}\cos^2\frac{\theta}{5}\sin^3\frac{\theta}{5}+5\cos\frac{\theta}{5}\sin^4\frac{\theta}{5}+\mathrm{i}\sin^5\frac{\theta}{5}$ A1

$\therefore \cos\theta+\mathrm{i}\sin\theta$

$= \cos^5\frac{\theta}{5}+5\mathrm{i}\cos^4\frac{\theta}{5}\sin\frac{\theta}{5}-10\cos^3\frac{\theta}{5}\sin^2\frac{\theta}{5}$ M1

$-10\mathrm{i}\cos^2\frac{\theta}{5}\sin^3\frac{\theta}{5}+5\cos\frac{\theta}{5}\sin^4\frac{\theta}{5}+\mathrm{i}\sin^5\frac{\theta}{5}$

$\cos\theta+\mathrm{i}\sin\theta$

$= \cos^5\frac{\theta}{5}-10\cos^3\frac{\theta}{5}\sin^2\frac{\theta}{5}+5\cos\frac{\theta}{5}\sin^4\frac{\theta}{5}$

$+\mathrm{i}\left(5\cos^4\frac{\theta}{5}\sin\frac{\theta}{5}-10\cos^2\frac{\theta}{5}\sin^3\frac{\theta}{5}+\sin^5\frac{\theta}{5}\right)$

$\therefore \cos\theta=\cos^5\frac{\theta}{5}-10\cos^3\frac{\theta}{5}\sin^2\frac{\theta}{5}+5\cos\frac{\theta}{5}\sin^4\frac{\theta}{5}$

and $\sin\theta=5\cos^4\frac{\theta}{5}\sin\frac{\theta}{5}-10\cos^2\frac{\theta}{5}\sin^3\frac{\theta}{5}+\sin^5\frac{\theta}{5}$. A2

[6]

(b) $\tan\theta = \dfrac{\sin\theta}{\cos\theta}$

$$\tan\theta = \frac{5\cos^4\frac{\theta}{5}\sin\frac{\theta}{5} - 10\cos^2\frac{\theta}{5}\sin^3\frac{\theta}{5} + \sin^5\frac{\theta}{5}}{\cos^5\frac{\theta}{5} - 10\cos^3\frac{\theta}{5}\sin^2\frac{\theta}{5} + 5\cos\frac{\theta}{5}\sin^4\frac{\theta}{5}}$$ M1A1

$$\tan\theta = \frac{5\tan\frac{\theta}{5} - 10\tan^3\frac{\theta}{5} + \tan^5\frac{\theta}{5}}{1 - 10\tan^2\frac{\theta}{5} + 5\tan^4\frac{\theta}{5}}$$ A1

$$\tan\theta = \frac{\tan\frac{\theta}{5}\left(5 - 10\tan^2\frac{\theta}{5} + \tan^4\frac{\theta}{5}\right)}{1 - 10\tan^2\frac{\theta}{5} + 5\tan^4\frac{\theta}{5}}$$ AG

[3]

(c) $x = \tan\dfrac{\theta}{5}$

$$\tan\theta = \frac{x(5 - 10x^2 + x^4)}{1 - 10x^2 + 5x^4}$$ M1

$x^4 - 10x^2 + 5 = 0$

$$\frac{x(5 - 10x^2 + x^4)}{1 - 10x^2 + 5x^4} = 0$$

$\tan\theta = 0$ M1

$\theta = 0,\ \pi,\ 2\pi,\ 3\pi$ or 4π

$\therefore x = \tan\dfrac{0}{5}$, $x = \tan\dfrac{\pi}{5}$, $x = \tan\dfrac{2\pi}{5}$, $x = \tan\dfrac{3\pi}{5}$ or

$x = \tan\dfrac{4\pi}{5}$ A1

$x = 0$ (*Rejected*), $x = \tan\dfrac{\pi}{5}$, $x = \tan\dfrac{2\pi}{5}$,

$x = \tan\dfrac{3\pi}{5}$ or $x = \tan\dfrac{4\pi}{5}$ A1

[4]

(d) (i) $\tan\frac{\pi}{5}+\tan\frac{2\pi}{5}+\tan\frac{3\pi}{5}+\tan\frac{4\pi}{5}=-\frac{0}{1}$ M1A1

$\tan\frac{0\pi}{5}+\tan\frac{\pi}{5}+\tan\frac{2\pi}{5}+\tan\frac{3\pi}{5}$

$+\tan\frac{4\pi}{5}=0$

$\therefore\sum_{r=0}^{4}\tan\frac{r\pi}{5}=0$ A1

(ii) $\left(\tan\frac{\pi}{5}\right)\left(\tan\frac{2\pi}{5}\right)\left(\tan\frac{3\pi}{5}\right)\left(\tan\frac{4\pi}{5}\right)=\frac{5}{1}$ M1A1

$\left(\tan\frac{\pi}{5}\right)\left(\tan\frac{2\pi}{5}\right)\left(\tan\left(\pi-\frac{2\pi}{5}\right)\right)$

$\left(\tan\left(\pi-\frac{\pi}{5}\right)\right)=5$

$\left(\tan\frac{\pi}{5}\right)\left(\tan\frac{2\pi}{5}\right)\left(-\tan\frac{2\pi}{5}\right)\left(-\tan\frac{\pi}{5}\right)=5$ A1

$\left(\tan\frac{\pi}{5}\tan\frac{2\pi}{5}\right)^2=5$

$\tan\frac{\pi}{5}\tan\frac{2\pi}{5}=\sqrt{5}$ A1

[7]

Chapter 10 Solution

Exercise 40

1. (a) $(x+2+h)^2=(x+2+h)(x+2+h)$

$(x+2+h)^2$

$=x^2+2x+hx+2x+4+2h+hx+2h+h^2$ (M1) for valid expansion

$(x+2+h)^2=x^2+(4+2h)x+(4+4h+h^2)$ A1

[2]

(b) $\frac{dy}{dx}=\lim_{h\to 0}\frac{(x+2+h)^2-(x+2)^2}{h}$

$\frac{dy}{dx}=\lim_{h\to 0}\frac{x^2+(4+2h)x+(4+4h+h^2)-(x^2+4x+4)}{h}$ (A1) for substitution

$\frac{dy}{dx}=\lim_{h\to 0}\frac{x^2+4x+2hx+4+4h+h^2-x^2-4x-4}{h}$

$\frac{dy}{dx}=\lim_{h\to 0}\frac{2hx+4h+h^2}{h}$

$\frac{dy}{dx}=\lim_{h\to 0}(2x+4+h)$ A1

$\frac{dy}{dx}=2x+4+0$

$\frac{dy}{dx}=2x+4$ A1

[3]

2. $f'(x)=\lim_{h\to 0}\frac{((x+h)^2-2(x+h)^3)-(x^2-2x^3)}{h}$ (A1) for substitution

$$f'(x)=\lim_{h\to 0}\frac{\begin{pmatrix}x^2+2hx+h^2\\ -2\left(x^3+\begin{pmatrix}3\\1\end{pmatrix}x^2h+\begin{pmatrix}3\\2\end{pmatrix}xh^2+h^3\right)\end{pmatrix}-(x^2-2x^3)}{h}$$ (M1) for valid expansion

$$f'(x)=\lim_{h\to 0}\frac{\begin{pmatrix}x^2+2hx+h^2\\ -2(x^3+3x^2h+3xh^2+h^3)\end{pmatrix}-x^2+2x^3}{h}$$

$$f'(x)=\lim_{h\to 0}\frac{\begin{matrix}x^2+2hx+h^2-2x^3-6x^2h\\ -6xh^2-2h^3-x^2+2x^3\end{matrix}}{h}$$

$$f'(x)=\lim_{h\to 0}\frac{2hx+h^2-6x^2h-6xh^2-2h^3}{h}$$ A1

$$f'(x)=\lim_{h\to 0}(2x+h-6x^2-6xh-2h^2)$$ A1

$$f'(x)=2x+0-6x^2-6x(0)-2(0)$$

$$f'(x)=2x-6x^2$$ A1

[5]

3. $$f'(x)=\lim_{h\to 0}\frac{\frac{1}{3(x+h)-2}-\frac{1}{3x-2}}{h}$$ (A1) for substitution

$$f'(x)=\lim_{h\to 0}\frac{\frac{1}{3x+3h-2}-\frac{1}{3x-2}}{h}$$

$$f'(x)=\lim_{h\to 0}\frac{1}{h}\left(\frac{3x-2}{(3x+3h-2)(3x-2)}-\frac{3x+3h-2}{(3x+3h-2)(3x-2)}\right)$$ (M1) for valid approach

$$f'(x)=\lim_{h\to 0}\frac{1}{h}\left(\frac{3x-2-3x-3h+2}{(3x+3h-2)(3x-2)}\right)$$

$$f'(x)=\lim_{h\to 0}\frac{1}{h}\left(\frac{-3h}{(3x+3h-2)(3x-2)}\right)$$ A1

$$f'(x)=\lim_{h\to 0}\left(\frac{-3}{(3x+3h-2)(3x-2)}\right)$$ A1

$$f'(x)=\frac{-3}{(3x+3(0)-2)(3x-2)}$$

$$f'(x)=-\frac{3}{(3x-2)^2}$$ A1

[5]

4. $$f'(x)=\lim_{h\to 0}\frac{\dfrac{1}{(1-2(x+h))^2}-\dfrac{1}{(1-2x)^2}}{h}$$ (A1) for substitution

$$f'(x)=\lim_{h\to 0}\frac{\dfrac{1}{(1-2x-2h)^2}-\dfrac{1}{(1-2x)^2}}{h}$$

$$f'(x)=\lim_{h\to 0}\frac{1}{h}\left(\frac{(1-2x)^2}{(1-2x-2h)^2(1-2x)^2}-\frac{(1-2x-2h)^2}{(1-2x-2h)^2(1-2x)^2}\right)$$ (M1) for valid approach

$$f'(x)=\lim_{h\to 0}\frac{1}{h}\left(\frac{(1-2x)^2-(1-2x-2h)^2}{(1-2x-2h)^2(1-2x)^2}\right)$$

$$f'(x)=\lim_{h\to 0}\frac{1}{h}\left(\frac{(1-2x+1-2x-2h)(1-2x-(1-2x-2h))}{(1-2x-2h)^2(1-2x)^2}\right)$$ A1

$$f'(x)=\lim_{h\to 0}\frac{1}{h}\left(\frac{(2-4x-2h)(2h)}{(1-2x-2h)^2(1-2x)^2}\right)$$

$$f'(x)=\lim_{h\to 0}\frac{4-8x-4h}{(1-2x-2h)^2(1-2x)^2}$$ A1

$$f'(x)=\frac{4(1-2x-0)}{(1-2x-2(0))^2(1-2x)^2}$$

$$f'(x)=\frac{4(1-2x)}{(1-2x)^4}$$

$$f'(x)=\frac{4}{(1-2x)^3}$$ A1

[5]

Exercise 41

1. Let $y = \arctan ex$.

$\tan y = ex$ A1

$\frac{\mathrm{d}}{\mathrm{d}x}(\tan y) = \frac{\mathrm{d}}{\mathrm{d}x}(ex)$

$\frac{\mathrm{d}(\tan y)}{\mathrm{d}y} \cdot \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}(ex)}{\mathrm{d}x}$ A1

$\sec^2 y \frac{\mathrm{d}y}{\mathrm{d}x} = e$ A1

$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{e}{\sec^2 y}$ M1

$\therefore \frac{\mathrm{d}}{\mathrm{d}x}(\arctan ex) = \frac{e}{1+\tan^2 y}$ A1

$\frac{\mathrm{d}}{\mathrm{d}x}(\arctan ex) = \frac{e}{1+(ex)^2}$

$\frac{\mathrm{d}}{\mathrm{d}x}(\arctan ex) = \frac{e}{1+e^2x^2}$ AG

[5]

2. Let $y = \arccos\frac{x}{3}$.

$\cos y = \frac{x}{3}$ A1

$\frac{\mathrm{d}}{\mathrm{d}x}(\cos y) = \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{x}{3}\right)$

$\frac{\mathrm{d}(\cos y)}{\mathrm{d}y} \cdot \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{x}{3}\right)$ A1

$-\sin y \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{3}$ A1

$\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{1}{3\sin y}$ M1

$\therefore \frac{\mathrm{d}}{\mathrm{d}x}\left(\arccos\frac{x}{3}\right) = -\frac{1}{3\sqrt{1-\cos^2 y}}$ A1

$\frac{\mathrm{d}}{\mathrm{d}x}\left(\arccos\frac{x}{3}\right) = -\frac{1}{3\sqrt{1-\left(\frac{x}{3}\right)^2}}$

$\frac{\mathrm{d}}{\mathrm{d}x}\left(\arccos\frac{x}{3}\right) = -\frac{1}{\sqrt{9\left(1-\frac{x^2}{9}\right)}}$

$\frac{\mathrm{d}}{\mathrm{d}x}\left(\arccos\frac{x}{3}\right) = -\frac{1}{\sqrt{9-x^2}}$ AG

[5]

3. Let $y = \arcsin x^3$.

$\sin y = x^3$ A1

$\frac{\text{d}}{\text{d}x}(\sin y) = \frac{\text{d}}{\text{d}x}(x^3)$

$\frac{\text{d}(\sin y)}{\text{d}y} \cdot \frac{\text{d}y}{\text{d}x} = \frac{\text{d}(x^3)}{\text{d}x}$ A1

$\cos y \frac{\text{d}y}{\text{d}x} = 3x^2$ A1

$\frac{\text{d}y}{\text{d}x} = \frac{3x^2}{\cos y}$ M1

$\therefore \frac{\text{d}}{\text{d}x}(\arcsin x^3) = \frac{3x^2}{\sqrt{1-\sin^2 y}}$ A1

$\frac{\text{d}}{\text{d}x}(\arcsin x^3) = \frac{3x^2}{\sqrt{1-(x^3)^2}}$

$\frac{\text{d}}{\text{d}x}(\arcsin x^3) = \frac{3x^2}{\sqrt{1-x^6}}$ AG

[5]

4. Let $y = \arctan \sqrt{x}$.

$\tan y = \sqrt{x}$ A1

$\frac{\text{d}}{\text{d}x}(\tan y) = \frac{\text{d}}{\text{d}x}(\sqrt{x})$

$\frac{\text{d}(\tan y)}{\text{d}y} \cdot \frac{\text{d}y}{\text{d}x} = \frac{\text{d}(\sqrt{x})}{\text{d}x}$ A1

$\sec^2 y \frac{\text{d}y}{\text{d}x} = \frac{1}{2\sqrt{x}}$ A1

$\frac{\text{d}y}{\text{d}x} = \frac{1}{2\sqrt{x}\sec^2 y}$ M1

$\therefore \frac{\text{d}}{\text{d}x}(\arctan \sqrt{x}) = \frac{1}{2\sqrt{x}(1+\tan^2 y)}$ A1

$\frac{\text{d}}{\text{d}x}(\arctan \sqrt{x}) = \frac{1}{2\sqrt{x}(1+(\sqrt{x})^2)}$

$\frac{\text{d}}{\text{d}x}(\arctan \sqrt{x}) = \frac{1}{2\sqrt{x}(1+x)}$ AG

[5]

Exercise 42

1. When $n=1$,

$$\text{L.H.S.}=-1\left(\frac{1}{(1+5x)^2}\right)(5)$$

$$\text{L.H.S.}=-\frac{5}{(1+5x)^2}$$

$$\text{R.H.S.}=\frac{(-5)^1 1!}{(1+5x)^{1+1}}$$

$$\text{R.H.S.}=-\frac{5}{(1+5x)^2}$$

Thus, the statement is true when $n=1$. R1

Assume that the statement is true when $n=k$. M1

$$f^{(k)}(x)=\frac{(-5)^k k!}{(1+5x)^{k+1}}$$

When $n=k+1$,

$$f^{(k+1)}(x)=\frac{\mathrm{d}}{\mathrm{d}x}(f^{(k)}(x))$$ M1

$$f^{(k+1)}(x)=(-5)^k k!(-(k+1))\left(\frac{1}{(1+5x)^{k+2}}\right)(5)$$ A1

$$f^{(k+1)}(x)=\frac{(-5)^k(-5)(k+1)k!}{(1+5x)^{k+2}}$$ A1

$$f^{(k+1)}(x)=\frac{(-5)^{k+1}(k+1)!}{(1+5x)^{k+2}}$$

$$f^{(k+1)}(x)=\frac{(-5)^{k+1}(k+1)!}{(1+5x)^{k+1+1}}$$ A1

Thus, the statement is true when $n=k+1$.

Therefore, the statement is true for all $n\in\mathbb{Z}^+$. R1

[7]

2. When $n=1$,

L.H.S. $=(2x)(e^x)+(x^2)(e^x)$

L.H.S. $=2xe^x+x^2e^x$

R.H.S. $=(1(1-1)+2(1)x+x^2)e^x$

R.H.S. $=2xe^x+x^2e^x$

Thus, the statement is true when $n=1$. R1

Assume that the statement is true when $n=k$. M1

$f^{(k)}(x)=(k(k-1)+2kx+x^2)e^x$

When $n=k+1$,

$f^{(k+1)}(x)=\frac{\mathrm{d}}{\mathrm{d}x}(f^{(k)}(x))$ M1

$f^{(k+1)}(x)=(0+2k+2x)(e^x)+(k(k-1)+2kx+x^2)(e^x)$ A1

$f^{(k+1)}(x)=(2k+2x+k^2-k+2kx+x^2)e^x$ A1

$f^{(k+1)}(x)=(k^2+k+2kx+2x+x^2)e^x$

$f^{(k+1)}(x)=((k+1)k+2(k+1)x+x^2)e^x$ A1

Thus, the statement is true when $n=k+1$.

Therefore, the statement is true for all $n\in\mathbb{Z}^+$. R1

[7]

3. When $n=2$,

$$\text{L.H.S.}=\frac{d}{dx}((1)(7^x)+(x)(7^x\ln 7))$$

$$\text{L.H.S.}=\frac{d}{dx}(7^x+x7^x\ln 7)$$

$$\text{L.H.S.}=7^x\ln 7+(1)(7^x\ln 7)+(x)(7^x(\ln 7)^2)$$

$$\text{L.H.S.}=2\cdot 7^x\ln 7+x7^x(\ln 7)^2$$

$$\text{R.H.S.}=2\cdot 7^x(\ln 7)^{2-1}+x7^x(\ln 7)^2$$

$$\text{R.H.S.}=2\cdot 7^x\ln 7+x7^x(\ln 7)^2$$

Thus, the statement is true when $n=2$. R1

Assume that the statement is true when $n=k$. M1

$$f^{(k)}(x)=k7^x(\ln 7)^{k-1}+x7^x(\ln 7)^k$$

When $n=k+1$,

$$f^{(k+1)}(x)=\frac{d}{dx}(f^{(k)}(x))$$ M1

$$f^{(k+1)}(x)=k7^x(\ln 7)^k+(1)(7^x(\ln 7)^k)+(x)(7^x(\ln 7)^{k+1})$$ A1

$$f^{(k+1)}(x)=k7^x(\ln 7)^k+7^x(\ln 7)^k+x7^x(\ln 7)^{k+1}$$ A1

$$f^{(k+1)}(x)=(k+1)7^x(\ln 7)^k+x7^x(\ln 7)^{k+1}$$

$$f^{(k+1)}(x)=(k+1)7^x(\ln 7)^{k+1-1}+x7^x(\ln 7)^{k+1}$$ A1

Thus, the statement is true when $n=k+1$.

Therefore, the statement is true for all $n\in\mathbb{Z}^+$, $n\geq 2$. R1

[7]

4. When $n=1$,

$$\text{L.H.S.}=-1\left(-\frac{1}{2}\right)(1+x)^{-\frac{3}{2}}(1)$$

$$\text{L.H.S.}=\frac{1}{2(1+x)^{\frac{3}{2}}}$$

$$\text{R.H.S.}=\frac{(-1)^{1+1}(2(1))!}{2^{2(1)}1!}(1+x)^{-1-\frac{1}{2}}$$

$$\text{R.H.S.}=\frac{1}{2(1+x)^{\frac{3}{2}}}$$

Thus, the statement is true when $n=1$. R1

Assume that the statement is true when $n=k$. M1

$$f^{(k)}(x)=\frac{(-1)^{k+1}(2k)!}{2^{2k}k!}(1+x)^{-k-\frac{1}{2}}$$

When $n=k+1$,

$$f^{(k+1)}(x)=\frac{\mathrm{d}}{\mathrm{d}x}(f^{(k)}(x))$$ M1

$$f^{(k+1)}(x)=\frac{(-1)^{k+1}(2k)!}{2^{2k}k!}\left(-k-\frac{1}{2}\right)(1+x)^{-k-\frac{3}{2}}(1)$$ A1

$$f^{(k+1)}(x)=\frac{(-1)^{k+1}(2k)!}{2^{2k}k!}\left(-\frac{2k+1}{2}\right)(1+x)^{-k-\frac{3}{2}}$$ A1

$$f^{(k+1)}(x)=\frac{(-1)^{k+2}(2k+1)(2k)!}{2^{2k+1}k!}(1+x)^{-k-\frac{3}{2}}$$

$$f^{(k+1)}(x)=\frac{(-1)^{k+2}(2k+2)(2k+1)(2k)!}{2^{2k+1}k!(2)(k+1)}(1+x)^{-k-\frac{3}{2}}$$ A1

$$f^{(k+1)}(x)=\frac{(-1)^{k+2}(2k+2)!}{2^{2k+2}(k+1)!}(1+x)^{-k-\frac{3}{2}}$$

$$f^{(k+1)}(x)=\frac{(-1)^{k+1+1}(2(k+1))!}{2^{2(k+1)}(k+1)!}(1+x)^{-(k+1)-\frac{1}{2}}$$ A1

Thus, the statement is true when $n=k+1$.

Therefore, the statement is true for all $n\in\mathbb{Z}^{+}$. R1

[8]

Exercise 43

1. $x + y - e^{2y} \ln x = 2$

$\frac{d}{dx}(x) + \frac{d}{dx}(y) - \frac{d}{dx}(e^{2y} \ln x) = \frac{d}{dx}(2)$ (M1) for valid approach

$1 + \frac{dy}{dx} - \left(\left(2e^{2y} \frac{dy}{dx} \right)(\ln x) + (e^{2y})\left(\frac{1}{x} \right) \right) = 0$ (A2) for correct approach

$1 + \frac{dy}{dx} - 2e^{2y} \ln x \frac{dy}{dx} - \frac{e^{2y}}{x} = 0$

$\frac{dy}{dx} - 2e^{2y} \ln x \frac{dy}{dx} = \frac{e^{2y}}{x} - 1$

$(1 - 2e^{2y} \ln x) \frac{dy}{dx} = \frac{e^{2y} - x}{x}$

$\frac{dy}{dx} = \frac{e^{2y} - x}{x(1 - 2e^{2y} \ln x)}$ A1

The required gradient

$= \frac{e^{2(1)} - 1}{1(1 - 2e^{2(1)} \ln 1)}$ (A1) for substitution

$= \frac{e^2 - 1}{1(1 - 0)}$

$= e^2 - 1$ A1

[6]

2. $(x-3)^2+4y^2=4(4y-3)$

$\frac{d}{dx}(x-3)^2+\frac{d}{dx}(4y^2)=\frac{d}{dx}(16y-12)$ (M1) for valid approach

$2(x-3)(1)+8y\frac{dy}{dx}=16\frac{dy}{dx}-0$ (A1) for correct approach

$2x-6+8y\frac{dy}{dx}=16\frac{dy}{dx}$

$x-3=8\frac{dy}{dx}-4y\frac{dy}{dx}$

$x-3=(8-4y)\frac{dy}{dx}$

$\frac{dy}{dx}=\frac{x-3}{8-4y}$ A1

$\therefore\frac{x-3}{8-4y}=0$ (A1) for substitution

$x-3=0$

$x=3$

$\therefore(3-3)^2+4y^2=4(4y-3)$ (M1) for substitution

$4y^2=4(4y-3)$

$y^2=4y-3$

$y^2-4y+3=0$ A1

$(y-1)(y-3)=0$

$y=1$ or $y=3$

Thus, the required coordinates are $(3,1)$ and $(3,3)$. A2

[8]

3. $xy + y^2 = 3$

$$\frac{\mathrm{d}}{\mathrm{d}x}(xy) + \frac{\mathrm{d}}{\mathrm{d}x}(y^2) = \frac{\mathrm{d}}{\mathrm{d}x}(3)$$ M1

$$(1)(y) + (x)\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right) + 2y\frac{\mathrm{d}y}{\mathrm{d}x} = 0$$ A2

$$y + x\frac{\mathrm{d}y}{\mathrm{d}x} + 2y\frac{\mathrm{d}y}{\mathrm{d}x} = 0$$

$$(x+2y)\frac{\mathrm{d}y}{\mathrm{d}x} = -y$$

$$\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{y}{x+2y}$$ A1

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = -\frac{(x+2y)\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right) - (y)\left(1 + 2\frac{\mathrm{d}y}{\mathrm{d}x}\right)}{(x+2y)^2}$$ A1

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = -\frac{x\frac{\mathrm{d}y}{\mathrm{d}x} - y}{(x+2y)^2}$$

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = \frac{y - x\frac{\mathrm{d}y}{\mathrm{d}x}}{(x+2y)^2}$$ A1

$$\therefore (x+2y)\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + \frac{\mathrm{d}y}{\mathrm{d}x} - \frac{xy}{(x+2y)^2}$$

$$= (x+2y)\left(\frac{y - x\frac{\mathrm{d}y}{\mathrm{d}x}}{(x+2y)^2}\right) + \frac{\mathrm{d}y}{\mathrm{d}x} - \frac{xy}{(x+2y)^2}$$ M1

$$= (x+2y)\left(\frac{y - x\left(-\frac{y}{x+2y}\right)}{(x+2y)^2}\right) - \frac{y}{x+2y} - \frac{xy}{(x+2y)^2}$$

$$= \frac{y}{x+2y} + \frac{xy}{(x+2y)^2} - \frac{y}{x+2y} - \frac{xy}{(x+2y)^2}$$

$$= 0$$ AG

[7]

4. $e^x + e^y + 4x = 0$

$\frac{d}{dx}(e^x) + \frac{d}{dx}(e^y) + \frac{d}{dx}(4x) = 0$ M1

$e^x + e^y \frac{dy}{dx} + 4 = 0$ A1

$e^y \frac{dy}{dx} = -(e^x + 4)$

$\frac{dy}{dx} = -\frac{e^x + 4}{e^y}$ A1

$\frac{d^2y}{dx^2} = -\frac{(e^y)(e^x + 0) - (e^x + 4)\left(e^y \frac{dy}{dx}\right)}{(e^y)^2}$ A1

$\frac{d^2y}{dx^2} = -\frac{e^{x+y} - (e^{x+y} + 4e^y)\frac{dy}{dx}}{e^{2y}}$

$\frac{d^2y}{dx^2} = (e^{x-y} + 4e^{-y})\frac{dy}{dx} - e^{x-y}$ A1

$\therefore e^{2y}\frac{d^2y}{dx^2} + (4 + e^x)^2 + e^{x+y}$

$= e^{2y}\left((e^{x-y} + 4e^{-y})\frac{dy}{dx} - e^{x-y}\right) + (4 + e^x)^2 + e^{x+y}$ M1

$= \left((e^{x+y} + 4e^y)\left(-\frac{e^x + 4}{e^y}\right) - e^{x+y}\right) + (4 + e^x)^2 + e^{x+y}$

$= -(e^x + 4)^2 - e^{x+y} + (4 + e^x)^2 + e^{x+y}$

$= 0$ AG

[6]

Exercise 44

1. (a) $e^x y = 12 - y^2$

$\frac{d}{dx}(e^x y) = \frac{d}{dx}(12) - \frac{d}{dx}(y^2)$ M1

$(e^x)(y) + (e^x)\left(\frac{dy}{dx}\right) = 0 - 2y\frac{dy}{dx}$ A2

$e^x y = -e^x\frac{dy}{dx} - 2y\frac{dy}{dx}$ M1

$e^x y = -(e^x + 2y)\frac{dy}{dx}$

$\frac{dy}{dx} = -\frac{e^x y}{e^x + 2y}$ AG

[4]

(b) (i) $3x + 7y - 21 = 0$

$7y = 21 - 3x$

$y = 3 - \frac{3}{7}x$ (M1) for valid approach

$\therefore e^x\left(3 - \frac{3}{7}x\right) = 12 - \left(3 - \frac{3}{7}x\right)^2$ (M1) for substitution

$\left(3 - \frac{3}{7}x\right)^2 + e^x\left(3 - \frac{3}{7}x\right) - 12 = 0$

By considering the graph of

$y = \left(3 - \frac{3}{7}x\right)^2 + e^x\left(3 - \frac{3}{7}x\right) - 12$,

$x = 0$ or $x = 6.9737895$ (*Rejected*). A1

$\therefore y = 3 - \frac{3}{7}(0)$

$y = 3$

Thus, the coordinates of P are $(0, 3)$. A1

(ii) The gradient of the tangent at P

$= -\frac{e^0(3)}{e^0 + 2(3)}$ (M1) for substitution

$= -\frac{3}{7}$ A1

[6]

(c) $e^0 y = 12 - y^2$

$y^2 + y - 12 = 0$ (A1) for correct equation

$(y+4)(y-3) = 0$

$y = -4$ or $y = 3$ (*Rejected*) A1

The gradient of the normal at Q

$= -1 \div -\dfrac{e^0(-4)}{e^0 + 2(-4)}$ (M1) for substitution

$= -1 \div -\dfrac{4}{7}$

$= \dfrac{7}{4}$ A1

[4]

2. (a) $2x^2 + axy + y^2 = 56$

$\frac{d}{dx}(2x^2) + \frac{d}{dx}(axy) + \frac{d}{dx}(y^2) = \frac{d}{dx}(56)$ M1

$4x + (a)(y) + (ax)\left(\frac{dy}{dx}\right) + 2y\frac{dy}{dx} = 0$ A2

$ax\frac{dy}{dx} + 2y\frac{dy}{dx} = -(4x + ay)$ M1

$(ax + 2y)\frac{dy}{dx} = -(4x + ay)$

$\frac{dy}{dx} = -\frac{4x + ay}{ax + 2y}$ AG

[4]

(b) (i) -1 A1

(ii) The gradient of the tangent at P

$= -1 \div -1$

$= 1$ (A1) for correct value

$\therefore -\frac{4(2) + a(-6)}{2a + 2(-6)} = 1$ (M1) for setting equation

$-\frac{8 - 6a}{2a - 12} = 1$

$6a - 8 = 2a - 12$ (M1) for valid approach

$4a = -4$

$a = -1$ A1

[5]

(c) $\frac{dy}{dx} = 0$

$-\frac{4x - y}{-x + 2y} = 0$ A1

$4x - y = 0 \Rightarrow y = 4x$ A1

$\therefore 2x^2 - x(4x) + (4x)^2 = 56$ M1

$2x^2 - 4x^2 + 16x^2 = 56$

$14x^2 = 56$

$x^2 = 4$

$x = -2$ or $x = 2$ A1

When $x = -2$, $y = 4(-2) = -8$. M1

When $x = 2$, $y = 4(2) = 8$.

Thus, that the coordinates of Q and R are $(2, 8)$ and $(-2, -8)$. AG

[5]

3. (a) $5x^2-2xy-3y^2-16x=0$

$\frac{d}{dx}(5x^2)-\frac{d}{dx}(2xy)-\frac{d}{dx}(3y^2)-\frac{d}{dx}(16x)=\frac{d}{dx}(0)$ M1

$10x-\left((2)(y)+(2x)\left(\frac{dy}{dx}\right)\right)-6y\frac{dy}{dx}-16=0$ A2

$10x-2y-2x\frac{dy}{dx}-6y\frac{dy}{dx}-16=0$

$5x-y-x\frac{dy}{dx}-3y\frac{dy}{dx}-8=0$

$x\frac{dy}{dx}+3y\frac{dy}{dx}=5x-y-8$ M1

$(x+3y)\frac{dy}{dx}=5x-y-8$

$\frac{dy}{dx}=\frac{5x-y-8}{x+3y}$ AG

[4]

(b) The gradient of the tangent

$=-1\div-\frac{1}{5}$

$=5$ (A1) for correct value

$\therefore \frac{5x-y-8}{x+3y}=5$ (M1) for setting equation

$5x-y-8=5(x+3y)$

$5x-y-8=5x+15y$

$16y=-8$

$y=-0.5$

$\therefore 5x^2-2x(-0.5)-3(-0.5)^2-16x=0$ (M1) for substitution

$5x^2+x-0.75-16x=0$

$5x^2-15x-0.75=0$

$20x^2-60x-3=0$ A1

By considering the graph of $y=20x^2-60x-3$,

$x=-0.049193$ or $x=3.0491933$.

Thus, that the coordinates of Q and R are

$(-0.0492,-0.5)$ and $(3.05,-0.5)$. A2

[6]

$$(x+3y)\frac{\mathrm{d}}{\mathrm{d}x}(5x-y-8)$$

(c) $$\frac{\mathrm{d}^2y}{\mathrm{d}x^2}=\frac{-(5x-y-8)\frac{\mathrm{d}}{\mathrm{d}x}(x+3y)}{(x+3y)^2}$$ (M1) for valid approach

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2}=\frac{(x+3y)\left(5-\frac{\mathrm{d}y}{\mathrm{d}x}-0\right)-(5x-y-8)\left(1+3\frac{\mathrm{d}y}{\mathrm{d}x}\right)}{(x+3y)^2}$$ (A1) for correct approach

$$5x+15y-(x+3y)\frac{\mathrm{d}y}{\mathrm{d}x}$$

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2}=\frac{-(5x-y-8)-(15x-3y-24)\frac{\mathrm{d}y}{\mathrm{d}x}}{(x+3y)^2}$$

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2}=\frac{16y+8+(-16x+24)\frac{\mathrm{d}y}{\mathrm{d}x}}{(x+3y)^2}$$ A1

$$\therefore (x+3y)^2\left(\frac{16y+8+(-16x+24)\frac{\mathrm{d}y}{\mathrm{d}x}}{(x+3y)^2}\right)$$ M1

$$\geq 8(3-2x)\frac{\mathrm{d}y}{\mathrm{d}x}$$

$$16y+8+(-16x+24)\frac{\mathrm{d}y}{\mathrm{d}x}\geq(24-16x)\frac{\mathrm{d}y}{\mathrm{d}x}$$ (M1) for valid approach

$$16y\geq -8$$

$$y\geq -\frac{1}{2}$$ A1

[6]

4. (a) $x + xy = 8 + y^2$

$\frac{d}{dx}(x) + \frac{d}{dx}(xy) = \frac{d}{dx}(8) + \frac{d}{dx}(y^2)$ M1

$1 + \left((1)(y) + (x)\left(\frac{dy}{dx}\right)\right) = 0 + 2y\frac{dy}{dx}$ A2

$1 + y + x\frac{dy}{dx} = 2y\frac{dy}{dx}$

$2y\frac{dy}{dx} - x\frac{dy}{dx} = 1 + y$ M1

$(2y - x)\frac{dy}{dx} = 1 + y$

$\frac{dy}{dx} = \frac{1+y}{2y-x}$ AG

[4]

(b) The gradient of the normal

$= -1 \div \frac{dy}{dx}$

$= -1 \div \frac{1+y}{2y-x}$

$= \frac{x-2y}{1+y}$ (A1) for correct approach

$\therefore \frac{x-2y}{1+y} = 0$ (M1) for setting equation

$x - 2y = 0$

$x = 2y$

$\therefore 2y + (2y)y = 8 + y^2$ (M1) for substitution

$2y + 2y^2 = 8 + y^2$

$y^2 + 2y - 8 = 0$ A1

$(y+4)(y-2) = 0$

$y = -4$ or $y = 2$ (A1) for correct values

When $y = -4$, $x = 2(-4) = -8$.

When $y = 2$, $x = 2(2) = 4$.

Thus, that the coordinates of P and Q are

$(-8, -4)$ and $(4, 2)$. A2

[7]

(c) $$\frac{d^2y}{dx^2}=\frac{(2y-x)\frac{d}{dx}(1+y)-(1+y)\frac{d}{dx}(2y-x)}{(2y-x)^2}$$ M1

$$\frac{d^2y}{dx^2}=\frac{(2y-x)\left(0+\frac{dy}{dx}\right)-(1+y)\left(2\frac{dy}{dx}-1\right)}{(2y-x)^2}$$ A1

$$\frac{d^2y}{dx^2}=\frac{(2y-x)\frac{dy}{dx}-(2+2y)\frac{dy}{dx}+1+y}{(2y-x)^2}$$ M1

$$\frac{d^2y}{dx^2}=\frac{(-2-x)\frac{dy}{dx}+1+y}{(2y-x)^2}$$ A1

$$\therefore \frac{(-2-x)\frac{dy}{dx}+1+y}{(2y-x)^2}-\frac{1}{1+y}\left(\frac{dy}{dx}\right)^2+\frac{(x+2)(y+1)}{(2y-x)^3}$$ M1

$$=\frac{(-2-x)\left(\frac{1+y}{2y-x}\right)+1+y}{(2y-x)^2}-\frac{1}{1+y}\left(\frac{1+y}{2y-x}\right)^2$$ M1

$$+\frac{(x+2)(y+1)}{(2y-x)^3}$$

$$=-\frac{(2+x)(1+y)}{(2y-x)^3}+\frac{1+y}{(2y-x)^2}$$

$$-\frac{1+y}{(2y-x)^2}+\frac{(x+2)(y+1)}{(2y-x)^3}$$

$$=0$$ AG

[6]

Chapter 11 Solution

Exercise 45

1. $y \ln x = x^2 - y$

$\frac{d}{dx}(y \ln x) = \frac{d}{dx}(x^2 - y)$ M1

$\left(\frac{dy}{dx}\right)(\ln x) + (y)\left(\frac{1}{x}\right) = 2x - \frac{dy}{dx}$ A2

$\ln x \frac{dy}{dx} + \frac{y}{x} = 2x - \frac{dy}{dx}$

$\ln x \frac{dy}{dx} + \frac{dy}{dx} = 2x - \frac{y}{x}$

$(\ln x + 1)\frac{dy}{dx} = 2x - \frac{y}{x}$

$\frac{dy}{dx} = \frac{2x - \frac{y}{x}}{\ln x + 1}$ A1

$\left.\frac{dy}{dx}\right|_{(1,1)} = \frac{2(1) - \frac{1}{1}}{\ln 1 + 1}$ M1

$\left.\frac{dy}{dx}\right|_{(1,1)} = \frac{2-1}{0+1}$

$\left.\frac{dy}{dx}\right|_{(1,1)} = 1$ A1

The equation of tangent:

$y - 1 = 1(x - 1)$ A1

$y - 1 = x - 1$

$y = x$ AG

[7]

2. $2y\arctan x=\frac{5}{4}x+\frac{\pi-5}{4}$

$\frac{d}{dx}(2y\arctan x)=\frac{d}{dx}\left(\frac{5}{4}x\right)+\frac{d}{dx}\left(\frac{\pi-5}{4}\right)$ (M1) for valid approach

$\left(2\frac{dy}{dx}\right)(\arctan x)+(2y)\left(\frac{1}{1+x^2}\right)=\frac{5}{4}-0$ (A2) for correct approach

$2\arctan x\frac{dy}{dx}+\frac{2y}{1+x^2}=\frac{5}{4}$

$2\arctan x\frac{dy}{dx}=\frac{5}{4}-\frac{2y}{1+x^2}$

$\frac{dy}{dx}=\frac{1}{2\arctan x}\left(\frac{5}{4}-\frac{2y}{1+x^2}\right)$

$\left.\frac{dy}{dx}\right|_{\left(1,\frac{1}{2}\right)}=\frac{1}{2\arctan 1}\left(\frac{5}{4}-\frac{2\left(\frac{1}{2}\right)}{1+1^2}\right)$ (M1) for substitution

$\left.\frac{dy}{dx}\right|_{\left(1,\frac{1}{2}\right)}=\frac{1}{2\left(\frac{\pi}{4}\right)}\left(\frac{5}{4}-\frac{1}{2}\right)$

$\left.\frac{dy}{dx}\right|_{\left(1,\frac{1}{2}\right)}=\frac{2}{\pi}\left(\frac{3}{4}\right)$

$\left.\frac{dy}{dx}\right|_{\left(1,\frac{1}{2}\right)}=\frac{3}{2\pi}$ A1

Thus, the slope of normal

$=-1\div\frac{3}{2\pi}$

$=-\frac{2\pi}{3}$ A1

The equation of normal:

$y-\frac{1}{2}=-\frac{2\pi}{3}(x-1)$ (A1) for substitution

$y-\frac{1}{2}=-\frac{2\pi}{3}x+\frac{2\pi}{3}$

$6y-3=-4\pi x+4\pi$

$4\pi x+6y-(4\pi+3)=0$ A1

[8]

3. $x^2 + 4y^2 = 100$

$\frac{d}{dx}(x^2) + \frac{d}{dx}(4y^2) = \frac{d}{dx}(100)$ (M1) for valid approach

$2x + 8y\frac{dy}{dx} = 0$ (A1) for correct approach

$8y\frac{dy}{dx} = -2x$

$\frac{dy}{dx} = -\frac{x}{4y}$ A1

$\therefore -\frac{x}{4y} = -\frac{3}{8}$ (M1) for setting equation

$x = \frac{3}{2}y$

$\therefore \left(\frac{3}{2}y\right)^2 + 4y^2 = 100$

$\frac{9}{4}y^2 + 4y^2 = 100$

$\frac{25}{4}y^2 = 100$

$y^2 = 16$

$y = -4$ or $y = 4$ A1

When $y = -4$, $x = \frac{3}{2}(-4) = -6$.

When $y = 4$, $x = \frac{3}{2}(4) = 6$.

The equations of tangent:

$y - (-4) = -\frac{3}{8}(x - (-6))$ or $y - 4 = -\frac{3}{8}(x - 6)$ (A1) for substitution

$8(y + 4) = -3(x + 6)$ or $8(y - 4) = -3(x - 6)$

$8y + 32 = -3x - 18$ or $8y - 32 = -3x + 18$

$3x + 8y + 50 = 0$ or $3x + 8y - 50 = 0$ A2

[8]

4. $9x^2 + y^2 = 106$

$\frac{d}{dx}(9x^2) + \frac{d}{dx}(y^2) = \frac{d}{dx}(106)$ (M1) for valid approach

$18x + 2y\frac{dy}{dx} = 0$ (A1) for correct approach

$2y\frac{dy}{dx} = -18x$

$\frac{dy}{dx} = -\frac{9x}{y}$ A1

$\therefore -\frac{9x}{y} \times \frac{5}{27} = -1$ (M1) for setting equation

$-\frac{5x}{3y} = -1$

$x = \frac{3}{5}y$

$\therefore 9\left(\frac{3}{5}y\right)^2 + y^2 = 106$

$\frac{81}{25}y^2 + y^2 = 106$

$\frac{106}{25}y^2 = 106$

$y^2 = 25$

$y = -5$ or $y = 5$ A1

When $y = -5$, $x = \frac{3}{5}(-5) = -3$.

When $y = 5$, $x = \frac{3}{5}(5) = 3$.

The equations of normal:

$y - (-5) = \frac{5}{27}(x - (-3))$ or $y - 5 = \frac{5}{27}(x - 3)$ (A1) for substitution

$27(y + 5) = 5(x + 3)$ or $27(y - 5) = 5(x - 3)$

$27y + 135 = 5x + 15$ or $27y - 135 = 5x - 15$

$5x - 27y - 120 = 0$ or $5x - 27y + 120 = 0$ A2

[8]

Exercise 46

1. (a) $f(a) = \arctan a$

$\arctan a + \ln(1+a^2) = \arctan a$ (M1) for setting equation

$\ln(1+a^2) = 0$

$1 + a^2 = 1$

$a^2 = 0$

$a = 0$ A1

[2]

(b) $f'(x) = \dfrac{1}{1+x^2} + \left(\dfrac{1}{1+x^2}\right)(2x)$ (A1) for correct approach

$f'(x) = \dfrac{1+2x}{1+x^2}$ A1

[2]

(c) $f'(x) = 0$

$\therefore \dfrac{1+2x}{1+x^2} = 0$ M1

$1 + 2x = 0$

$2x = -1$

$x = -\dfrac{1}{2}$

By the first derivative test, M1A1

x	$x < -\frac{1}{2}$	$x = -\frac{1}{2}$	$x > -\frac{1}{2}$
$f'(x)$	–	0	+

Thus, there is no local maximum of $f(x)$ for $x \in \mathbb{R}$. AG

[3]

(d) By the first derivative test, $f(x)$ attains its minimum at $x=-\frac{1}{2}$. (R1) for correct argument

$$f\left(-\frac{1}{2}\right)=\arctan\left(-\frac{1}{2}\right)+\ln\left(1+\left(-\frac{1}{2}\right)^2\right)$$

$$f\left(-\frac{1}{2}\right)=\arctan\left(-\frac{1}{2}\right)+\ln\frac{5}{4}$$

Thus, the coordinates of the local minimum of $f(x)$ are $\left(-\frac{1}{2},\arctan\left(-\frac{1}{2}\right)+\ln\frac{5}{4}\right)$. A1

[2]

(e) $$f''(x)=\frac{(1+x^2)(2)-(1+2x)(2x)}{(1+x^2)^2}$$ (A1) for correct approach

$$f''(x)=\frac{2+2x^2-2x-4x^2}{(1+x^2)^2}$$

$$f''(x)=\frac{2-2x-2x^2}{(1+x^2)^2}$$

$$f''(x)=\frac{2(1-x-x^2)}{(1+x^2)^2}$$ A1

[2]

(f) $f''(x)=0$

$\therefore \dfrac{2(1-x-x^2)}{(1+x^2)^2}=0$ M1

$2(1-x-x^2)=0$

$x^2+x-1=0$ A1

$x=\dfrac{-1\pm\sqrt{1^2-4(1)(-1)}}{2(1)}$

$x=\dfrac{-1+\sqrt{5}}{2}$ or $x=\dfrac{-1-\sqrt{5}}{2}$ A1

x	$x<\dfrac{-1-\sqrt{5}}{2}$	$x=\dfrac{-1-\sqrt{5}}{2}$	$\dfrac{-1-\sqrt{5}}{2}<x<\dfrac{-1+\sqrt{5}}{2}$	$x=\dfrac{-1+\sqrt{5}}{2}$	$x>\dfrac{-1+\sqrt{5}}{2}$
$f''(x)$	–	0	+	0	–

$f''(x)$ changes its sign at $x=\dfrac{-1-\sqrt{5}}{2}$ and

$x=\dfrac{-1+\sqrt{5}}{2}$. M1

Thus, there are two points of inflexion of $f(x)$ for

$x\in\mathbb{R}$. AG

[4]

(g) $f(-x)=0$ has the solutions $x=0$ and $x=1.17$. (A1) for correct values

The x-coordinate of the local minimum of $f(-x)$

is $\dfrac{1}{2}$.

$\therefore x\le 0$ or $x\ge 1.17$. A2

[3]

2. (a) $y=2$ A1

[1]

(b) $f'(x)=0-e^{-\frac{1}{2}x^2}(-x)$ (A1) for correct approach

$f'(x)=xe^{-\frac{1}{2}x^2}$ A1

[2]

(c) $f'(x)=0$

$\therefore xe^{-\frac{1}{2}x^2}=0$ M1

$x=0$

By the first derivative test, M1A1

x	$x<0$	$x=0$	$x>0$
$f'(x)$	–	0	+

Thus, there is no local maximum of $f(x)$ for $x\in\mathbb{R}$. AG

[3]

(d) By the first derivative test, $f(x)$ attains its minimum at $x=0$. (R1) for correct argument

$f(0)=2-e^{-\frac{1}{2}(0)^2}$ (M1) for substitution

$f(0)=2-1$

$f(0)=1$

Thus, the range of $f(x)$ is $1\le y<2$. A1

[3]

(e) $f''(x)=(1)\left(e^{-\frac{1}{2}x^2}\right)+(x)\left(e^{-\frac{1}{2}x^2}\right)(-x)$ (A1) for correct approach

$f''(x)=(1-x^2)e^{-\frac{1}{2}x^2}$ A1

[2]

(f) $f''(x)=0$

$\therefore (1-x^2)e^{-\frac{1}{2}x^2}=0$ M1

$1-x^2=0$

$x^2=1$

$x=-1$ or $x=1$ A1

x	$x<-1$	$x=-1$	$-1<x<1$	$x=1$	$x>1$
$f''(x)$	–	0	+	0	–

$f''(x)$ changes its sign at $x=-1$ and $x=1$. M1

Thus, there are two points of inflexion of $f(x)$ for $x\in\mathbb{R}$. AG

[3]

(g) The y-coordinate of the points of inflexion of

$f(x)$

$= 2 - e^{-\frac{1}{2}(1)^2}$ (M1) for substitution

$= 2 - e^{-\frac{1}{2}}$

The y-coordinate of the points of inflexion of

$g(x) = \sqrt{e} f(x) + k$

$= \sqrt{e}\left(2 - e^{-\frac{1}{2}}\right) + k$ (A1) for correct approach

$= 2\sqrt{e} - 1 + k$

$\therefore 2\sqrt{e} - 1 + k = 0$

$k = 1 - 2\sqrt{e}$ A1

[3]

3. (a) $x=-6,\ x=6$ A2

[2]

(b) $f'(x)=\dfrac{(x^2-36)(1)-(x)(2x)}{(x^2-36)^2}$ (A1) for correct approach

$f'(x)=\dfrac{x^2-36-2x^2}{(x^2-36)^2}$

$f'(x)=-\dfrac{36+x^2}{(x^2-36)^2}$ A1

[2]

(c) $f'(x)=0$

$\therefore -\dfrac{36+x^2}{(x^2-36)^2}=0$ M1

$36+x^2=0$

$x^2=-36$

$\therefore f'(x)\neq 0$ for $x\neq -6,\ x\neq 6$. A1

Thus, there is no local extrema of $f(x)$. AG

[2]

(d) $f''(x)=-\dfrac{(x^2-36)^2(2x)-(36+x^2)(2)(x^2-36)(2x)}{((x^2-36)^2)^2}$ (A1) for correct approach

$f''(x)=-\dfrac{2x(x^2-36)^2-4x(36+x^2)(x^2-36)}{((x^2-36)^2)^2}$

$f''(x)=\dfrac{-2x(x^2-36)+4x(36+x^2)}{(x^2-36)^3}$ (M1) for simplification

$f''(x)=\dfrac{-2x^3+72x+144x+4x^3}{(x^2-36)^3}$

$f''(x)=\dfrac{2x^3+216x}{(x^2-36)^3}$

$f''(x)=\dfrac{2x(x^2+108)}{(x^2-36)^3}$ A1

[3]

(e) $f''(x)=0$

$\therefore \dfrac{2x(x^2+108)}{(x^2-36)^3}=0$ M1

$2x(x^2+108)=0$

$x=0$ A1

x	$x<0$	$x=0$	$x>0$
$f''(x)$	–	0	+

$f''(x)$ changes its sign at $x=0$ only. M1

Thus, there is only one point of inflexion of $f(x)$. AG

[3]

(f) Let $\dfrac{x}{x^2-36}\equiv\dfrac{A}{x+6}+\dfrac{B}{x-6}$, where A and B are constants.

$\dfrac{x}{x^2-36}\equiv\dfrac{A(x-6)}{(x+6)(x-6)}+\dfrac{B(x+6)}{(x+6)(x-6)}$ M1

$\dfrac{x}{x^2-36}\equiv\dfrac{Ax-6A+Bx+6B}{(x+6)(x-6)}$

$x\equiv(A+B)x+(-6A+6B)$ A1

$1=A+B$

$B=1-A$

$0=-6A+6B$

$\therefore 0=-6A+6(1-A)$ A1

$12A=6$

$A=\dfrac{1}{2}$

$\therefore B=1-\dfrac{1}{2}$

$B=\dfrac{1}{2}$

$\therefore \dfrac{x}{x^2-36}\equiv\dfrac{1}{2(x+6)}+\dfrac{1}{2(x-6)}$ A1

$f(x)=\dfrac{1}{2(x+12-6)}+\dfrac{1}{2(x-6)}$

$f(x)=g(x+12)+g(x)$

$\therefore g(x)=\dfrac{1}{2(x-6)}$ A1

[5]

4. (a) $x=-5,\ x=5$ A2

[2]

(b) $f'(x)=\dfrac{(x^2-25)(2x)-(x^2+25)(2x)}{(x^2-25)^2}$ (A1) for correct approach

$f'(x)=\dfrac{2x^3-50x-2x^3-50x}{(x^2-25)^2}$

$f'(x)=-\dfrac{100x}{(x^2-25)^2}$ A1

[2]

(c) $f'(x)=0$

$\therefore -\dfrac{100x}{(x^2-25)^2}=0$ M1

$100x=0$

$x=0$

By the first derivative test, M1A1

x	$x<-5$	$x=-5$	$-5<x<0$	$x=0$	$0<x<5$	$x=5$	$x>5$
$f'(x)$	+	Undefined	+	0	–	Undefined	–

Thus, there is no local minimum of $f(x)$. AG

[3]

(d) $f''(x)=-\dfrac{(x^2-25)^2(100)-(100x)(2)(x^2-25)(2x)}{((x^2-25)^2)^2}$ (A1) for correct approach

$f''(x)=\dfrac{-100(x^2-25)^2+400x^2(x^2-25)}{(x^2-25)^4}$

$f''(x)=\dfrac{-100(x^2-25)+400x^2}{(x^2-25)^3}$ (M1) for simplification

$f''(x)=\dfrac{-100x^2+2500+400x^2}{(x^2-25)^3}$

$f''(x)=\dfrac{100(3x^2+25)}{(x^2-25)^3}$ A1

[3]

(e) $f''(x)=0$

$\therefore \dfrac{100(3x^2+25)}{(x^2-25)^3}=0$ M1

$100(3x^2+25)=0$

$3x^2+25=0$

$x^2=-\dfrac{25}{3}$ A1

Therefore, $f''(x)\neq 0$ for $x\neq -5$, $x\neq 5$.

Thus, there is no point of inflexion of $f(x)$. AG

[2]

(f) (i) $f(x)\equiv \dfrac{A}{x+5}+\dfrac{B}{x-5}+C$

$\dfrac{x^2+25}{x^2-25}\equiv \dfrac{A(x-5)}{(x+5)(x-5)}$ M1A1

$+\dfrac{B(x+5)}{(x+5)(x-5)}+\dfrac{C(x^2-25)}{x^2-25}$

$\dfrac{x^2+25}{x^2-25}\equiv \dfrac{Ax-5A+Bx+5B+Cx^2-25C}{(x+5)(x-5)}$

$x^2+25\equiv Cx^2+(A+B)x+(-5A+5B-25C)$ A1

$C=1$ A1

$0=A+B$

$B=-A$

$25=-5A+5B-25C$

$\therefore 25=-5A+5(-A)-25(1)$

$10A=-50$

$A=-5$ A1

$\therefore B=-(-5)$

$B=5$ A1

(ii) $y=1$ A1

[7]

Exercise 47

1. (a) $V=\left(\frac{1}{2}r^2\theta\right)h$ A1

$\therefore 50=\frac{1}{2}r^2\theta h$ M1

$h=\frac{100}{r^2\theta}$ AG

[2]

(b) $A=2\left(\frac{1}{2}r^2\theta\right)+(r\theta)(h)+2(rh)$ (M1) for valid approach

$A=r^2\theta+r\theta h+2rh$

$A=r^2\theta+r\theta\left(\frac{100}{r^2\theta}\right)+2r\left(\frac{100}{r^2\theta}\right)$ (M1) for substitution

$A=r^2\theta+\frac{100}{r}+\frac{200}{r\theta}$ A1

[3]

(c) $r = r\theta$

$\theta = 1$ A1

$\therefore A = r^2(1) + \dfrac{100}{r} + \dfrac{200}{r(1)}$

$A = r^2 + \dfrac{300}{r}$ A1

$\dfrac{dA}{dr} = 2r + 300(-r^{-2})$ A1

$\dfrac{dA}{dr} = 2r - \dfrac{300}{r^2}$

$\dfrac{dA}{dr} = 0$

$\therefore 2r - \dfrac{300}{r^2} = 0$ M1

$2r^3 - 300 = 0$

$r^3 = 150$

$r = \sqrt[3]{150}$ A1

By the first derivative test, M1A1

x	$0 < x < \sqrt[3]{150}$	$x = \sqrt[3]{150}$	$x > \sqrt[3]{150}$
$\dfrac{dC}{dx}$	–	0	+

Thus, A attains its minimum when $r = \sqrt[3]{150}$. AG

[7]

(d) $(\sqrt[3]{150})^2 + \dfrac{300}{\sqrt[3]{150}} = k\sqrt[3]{22500}$ (M1) for setting equation

$\sqrt[3]{150^2} + \dfrac{300}{\sqrt[3]{150}} = k\sqrt[3]{150^2}$

$150 + 300 = k(150)$ (A1) for correct approach

$450 = 150k$

$k = 3$ A1

[3]

(e) $h = \dfrac{100}{(\sqrt[3]{150})^2(1)}$ (M1) for substitution

$h = \dfrac{100}{\sqrt[3]{150^2}}$

$h = \dfrac{100\sqrt[3]{150}}{150}$ (A1) for correct approach

$h = \dfrac{2}{3}\sqrt[3]{150}$

$h = \dfrac{2}{3}r$

$\therefore b = \dfrac{2}{3}$ A1

[3]

2. (a) $A = 2\pi r^2 + 2\pi rh + \pi r^2$ A1

$A = 3\pi r^2 + 2\pi rh$

$\therefore 125 = 3\pi r^2 + 2\pi rh$ M1

$\frac{125}{2\pi r} = \frac{3}{2}r + h$

$h = \frac{125}{2\pi r} - \frac{3}{2}r$ AG

[2]

(b) $h : r = 11:1$

$\therefore \frac{h}{r} = \frac{11}{1}$

$h = 11r$ (A1) for correct approach

$h = \frac{125}{2\pi r} - \frac{3}{2}r$

$\therefore 11r = \frac{125}{2\pi r} - \frac{3}{2}r$ (M1) for substitution

$11r^2 = \frac{125}{2\pi} - \frac{3}{2}r^2$

$\frac{25}{2}r^2 = \frac{125}{2\pi}$ (A1) for correct approach

$r^2 = \frac{5}{\pi}$

$r = \sqrt{\frac{5}{\pi}}$ A1

[4]

(c) $V = \frac{2}{3}\pi r^3 + \pi r^2 h$ (M1) for valid approach

$V = \frac{2}{3}\pi r^3 + \pi r^2\left(\frac{125}{2\pi r} - \frac{3}{2}r\right)$ (M1) for substitution

$V = \frac{2}{3}\pi r^3 + \frac{125}{2}r - \frac{3}{2}\pi r^3$

$V = \frac{125}{2}r - \frac{5}{6}\pi r^3$ A1

[3]

(d) $V = \frac{125}{2}r - \frac{5}{6}\pi r^3$

$\frac{dV}{dr} = \frac{125}{2}(1) - \frac{5}{6}\pi(3r^2)$ A1

$\frac{dV}{dr} = \frac{125}{2} - \frac{5}{2}\pi r^2$

$\frac{dV}{dr} = 0$

$\therefore \frac{125}{2} - \frac{5}{2}\pi r^2 = 0$ M1

$125 - 5\pi r^2 = 0$

$\pi r^2 = 25$

$r = \sqrt{\frac{25}{\pi}}$

$r = \frac{5}{\sqrt{\pi}}$ A1

By the first derivative test, M1A1

r	$0 < r < \frac{5}{\sqrt{\pi}}$	$r = \frac{5}{\sqrt{\pi}}$	$r > \frac{5}{\sqrt{\pi}}$
$\frac{dV}{dr}$	+	0	–

Thus, V attains its maximum when $r = \frac{5}{\sqrt{\pi}}$. AG

[5]

(e) $\frac{125}{2}\left(\frac{5}{\sqrt{\pi}}\right) - \frac{5}{6}\pi\left(\frac{5}{\sqrt{\pi}}\right)^3 = \sqrt{\frac{25}{k\pi}}$ (M1) for setting equation

$\frac{625}{2\sqrt{\pi}} - \frac{625}{6\sqrt{\pi}} = \frac{625}{\sqrt{k\pi}}$

$\frac{625}{3\sqrt{\pi}} = \frac{625}{\sqrt{k\pi}}$ (A1) for correct approach

$\because 3\sqrt{\pi} = \sqrt{k\pi}$ (M1) for valid approach

$9\pi = k\pi$

$k = 9$ A1

[4]

3. (a) $R^2 = \text{OP}^2 + r^2$ (M1) for valid approach

$\text{OP} = \sqrt{R^2 - r^2}$

$V = \frac{1}{3}\pi r^2 h$

$\therefore V = \frac{1}{3}\pi r^2 (R + \text{OP})$ (A1) for substitution

$\therefore V = \frac{1}{3}\pi r^2 (R + \sqrt{R^2 - r^2})$ A1

[3]

(b) $V = \frac{1}{3}\pi r^2 (R + \sqrt{R^2 - r^2})$

$\frac{\mathrm{d}V}{\mathrm{d}r} = \frac{1}{3}\pi(2r)(R + \sqrt{R^2 - r^2})$

$+\frac{1}{3}\pi r^2 \left(\frac{1}{2\sqrt{R^2 - r^2}}\right)(-2r)$ M1A1

$\frac{\mathrm{d}V}{\mathrm{d}r} = \frac{2}{3}\pi r(R + \sqrt{R^2 - r^2}) - \frac{1}{3}\pi r^3 \left(\frac{1}{\sqrt{R^2 - r^2}}\right)$ A1

$\frac{\mathrm{d}V}{\mathrm{d}r} = \frac{2\pi r(R\sqrt{R^2 - r^2} + R^2 - r^2)}{3\sqrt{R^2 - r^2}} - \frac{\pi r^3}{3\sqrt{R^2 - r^2}}$ M1

$\frac{\mathrm{d}V}{\mathrm{d}r} = \frac{\pi r(2R\sqrt{R^2 - r^2} + 2R^2 - 2r^2 - r^2)}{3\sqrt{R^2 - r^2}}$ A1

$\frac{\mathrm{d}V}{\mathrm{d}r} = \frac{\pi r(2R\sqrt{R^2 - r^2} + 2R^2 - 3r^2)}{3\sqrt{R^2 - r^2}}$ AG

[5]

(c) $\frac{dV}{dr}=0$

$\therefore \frac{\pi r(2R\sqrt{R^2-r^2}+2R^2-3r^2)}{3\sqrt{R^2-r^2}}=0$ M1

$2R\sqrt{R^2-r^2}+2R^2-3r^2=0$

$2R\sqrt{R^2-r^2}=3r^2-2R^2$

$\therefore 4R^2(R^2-r^2)=(3r^2-2R^2)^2$ M1

$4R^4-4R^2r^2=9r^4-12R^2r^2+4R^4$ A1

$8R^2r^2=9r^4$

$r^2=\frac{8}{9}R^2$

$r=\frac{2\sqrt{2}}{3}R$ A1

By the first derivative test, M1A1

r	$0<r<\frac{2\sqrt{2}}{3}R$	$r=\frac{2\sqrt{2}}{3}R$	$\frac{2\sqrt{2}}{3}R<r<R$
$\frac{dV}{dr}$	+	0	−

Thus, V attains its maximum when $r=\frac{2\sqrt{2}}{3}R$. AG

[6]

(d) $\frac{1}{3}\pi\left(\frac{8}{9}R^2\right)\left(R+\sqrt{R^2-\frac{8}{9}R^2}\right)=\frac{k}{81}\pi R^3$ (M1) for setting equation

$\frac{8}{27}\pi R^2\left(R+\sqrt{\frac{1}{9}R^2}\right)=\frac{k}{81}\pi R^3$

$\frac{8}{27}\pi R^2\left(R+\frac{1}{3}R\right)=\frac{k}{81}\pi R^3$ (A1) for correct approach

$\frac{8}{27}\pi R^2\left(\frac{4}{3}R\right)=\frac{k}{81}\pi R^3$

$\frac{32}{81}\pi R^3=\frac{k}{81}\pi R^3$ (A1) for correct approach

$k=32$ A1

[4]

(e) The minimum capacity

$= \frac{4}{3}\pi R^3 - \frac{32}{81}\pi R^3$ (M1) for valid approach

$= \frac{76}{81}\pi R^3$ A1

[2]

4. (a) Let l be the distance between the centre of the top square face and one of its vertices.

$l^2 + l^2 = x^2$ M1

$2l^2 = x^2$

$\sqrt{2}l = x$

$l = \frac{x}{\sqrt{2}}$ A1

By considering a pair of similar triangles,

$\frac{H-y}{l} = \frac{H}{R}$ M1

$H - y = \frac{H}{R}l$

$y = H - \frac{H}{R}l$

$y = H - \frac{H}{R}\left(\frac{x}{\sqrt{2}}\right)$ M1

$y = H - \frac{H}{\sqrt{2}R}x$

$y = H - \frac{\sqrt{2}H}{2R}x$ AG

[4]

(b) $V = x^2 y$

$\therefore V = x^2\left(H - \frac{\sqrt{2}H}{2R}x\right)$ (M1) for substitution

$V = Hx^2 - \frac{\sqrt{2}H}{2R}x^3$ A1

[2]

(c) $\frac{dV}{dx} = H(2x) - \frac{\sqrt{2}H}{2R}(3x^2)$ A1

$\frac{dV}{dx} = 2Hx - \frac{3\sqrt{2}H}{2R}x^2$ A1

$\frac{dV}{dx} = 0$

$\therefore 2Hx - \frac{3\sqrt{2}H}{2R}x^2 = 0$ M1

$4HRx - 3\sqrt{2}Hx^2 = 0$

$4HRx = 3\sqrt{2}Hx^2$

$x = \frac{4}{3\sqrt{2}}R$

$x = \frac{2\sqrt{2}}{3}R$ A1

By the first derivative test, M1A1

x	$0 < x < \frac{2\sqrt{2}}{3}R$	$x = \frac{2\sqrt{2}}{3}R$	$\frac{2\sqrt{2}}{3}R < x < R$
$\frac{dV}{dx}$	+	0	−

Thus, V attains its maximum when $x = \frac{2\sqrt{2}}{3}R$. AG

[6]

(d) The maximum value of V

$= H\left(\frac{2\sqrt{2}}{3}R\right)^2 - \frac{\sqrt{2}H}{2R}\left(\frac{2\sqrt{2}}{3}R\right)^3$ (M1) for substitution

$= H\left(\frac{8}{9}R^2\right) - \frac{\sqrt{2}H}{2R}\left(\frac{16\sqrt{2}}{27}R^3\right)$ (A1) for correct approach

$= \frac{8}{9}HR^2 - \frac{16}{27}HR^2$

$= \frac{8}{27}HR^2$ A1

[3]

(e) The total surface area

$= 2x^2 + 4xy$ (M1) for valid approach

$= 2\left(\frac{2\sqrt{2}}{3}R\right)^2 + 4\left(\frac{2\sqrt{2}}{3}R\right)\left(H - \frac{\sqrt{2}H}{2R}\left(\frac{2\sqrt{2}}{3}R\right)\right)$ (A1) for substitution

$= 2\left(\frac{8}{9}R^2\right) + 4\left(\frac{2\sqrt{2}}{3}R\right)\left(H - \frac{2H}{3}\right)$ (A1) for correct approach

$= \frac{16}{9}R^2 + \frac{4H}{3}\left(\frac{2\sqrt{2}}{3}R\right)$

$= \frac{16}{9}R^2 + \frac{8\sqrt{2}}{9}HR$

$= \frac{8}{9}R(2R + \sqrt{2}H)$ A1

[4]

Exercise 48

1. (a) $x^2 + y^2 = 50^2$

$\frac{d}{dt}(x^2) + \frac{d}{dt}(y^2) = \frac{d}{dt}(2500)$ (M1) for valid approach

$2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 0$ (A1) for correct approach

$x^2 + 14^2 = 50^2$

$x^2 = 2304$

$x = 48$

$\therefore 2(48)\frac{dx}{dt} + 2(14)(10) = 0$ A1

$96\frac{dx}{dt} = -280$

$\frac{dx}{dt} = -\frac{35}{12}\ \text{ms}^{-1}$ A1

[4]

(b) Let $\theta = \text{O}\hat{\text{B}}\text{A}$.

$\sin\theta = \frac{y}{50}$

$\frac{d}{dt}(\sin\theta) = \frac{d}{dt}\left(\frac{y}{50}\right)$

$\cos\theta\frac{d\theta}{dt} = \frac{1}{50}\frac{dy}{dt}$ (A1) for correct approach

$\sin\theta = \frac{14}{50}$

$\therefore \sqrt{1-\left(\frac{14}{50}\right)^2}\frac{d\theta}{dt} = \frac{1}{50}(10)$ (A1) for substitution

$\frac{24}{25}\frac{d\theta}{dt} = \frac{1}{5}$

$\frac{d\theta}{dt} = \frac{5}{24}\ \text{rads}^{-1}$ A1

[3]

2.	(a)	$x^2 + y^2 = H^2$	A1
		$\frac{\mathrm{d}}{\mathrm{d}t}(x^2) + \frac{\mathrm{d}}{\mathrm{d}t}(y^2) = \frac{\mathrm{d}}{\mathrm{d}t}(H^2)$	M1
		$2x\frac{\mathrm{d}x}{\mathrm{d}t} + 2y\frac{\mathrm{d}y}{\mathrm{d}t} = 2H\frac{\mathrm{d}H}{\mathrm{d}t}$	A1
		$\therefore 2x(40) + 2y(9) = 2\sqrt{x^2 + y^2}\frac{\mathrm{d}H}{\mathrm{d}t}$	A1
		$40x + 9y = \sqrt{x^2 + y^2}\frac{\mathrm{d}H}{\mathrm{d}t}$	
		$\frac{\mathrm{d}H}{\mathrm{d}t} = \frac{40x + 9y}{\sqrt{x^2 + y^2}}$	AG
			[4]
	(b)	$40 = \frac{x}{t} \Rightarrow x = 40t$	(M1) for valid approach
		$9 = \frac{y}{t} \Rightarrow y = 9t$	
		$\therefore \frac{\mathrm{d}H}{\mathrm{d}t} = \frac{40(40t) + 9(9t)}{\sqrt{(40t)^2 + (9t)^2}}$	(A1) for correct approach
		$\frac{\mathrm{d}H}{\mathrm{d}t} = \frac{1681t}{41t}$	
		$\frac{\mathrm{d}H}{\mathrm{d}t} = 41$	A1
			[3]

3. $z = xy^2 + \ln y$

$\frac{d}{dt}(z) = \frac{d}{dt}(xy^2) + \frac{d}{dt}(\ln y)$ (M1) for valid approach

$\frac{dz}{dt} = \left(\frac{dx}{dt}\right)(y^2) + (x)\left(2y\frac{dy}{dt}\right) + \frac{1}{y}\frac{dy}{dt}$ (A2) for correct approach

$\frac{dz}{dt} = y^2\frac{dx}{dt} + \left(2xy + \frac{1}{y}\right)\frac{dy}{dt}$

$5 = xe^2 + \ln e$ (M1) for substitution

$4 = xe^2$

$x = \frac{4}{e^2}$

$\therefore \frac{dz}{dt} = e^2\left(-\frac{5}{e}\right) + \left(2\left(\frac{4}{e^2}\right)e + \frac{1}{e}\right)(e^2)$ (A1) for substitution

$\frac{dz}{dt} = -5e + \left(\frac{9}{e}\right)(e^2)$

$\frac{dz}{dt} = 4e$ A1

[6]

4. $\dfrac{\sin\beta}{x} = \dfrac{\sin\frac{\pi}{6}}{60}$ (M1)(A1) for substitution

$\dfrac{\sin\beta}{x} = \dfrac{1}{120}$

$x = 120\sin\beta$ A1

$\dfrac{d}{dt}(x) = \dfrac{d}{dt}(120\sin\beta)$

$\dfrac{dx}{dt} = 120\cos\beta\dfrac{d\beta}{dt}$ A1

$\dfrac{\sin\beta}{60\sqrt{2}} = \dfrac{\sin\frac{\pi}{6}}{60}$

$\sin\beta = \dfrac{\sqrt{2}}{2}$

$\beta = \dfrac{\pi}{4}$ (A1) for correct value

$\therefore 90\sqrt{2} = 120\cos\dfrac{\pi}{4}\dfrac{d\beta}{dt}$ (A1) for substitution

$90\sqrt{2} = 120\left(\dfrac{\sqrt{2}}{2}\right)\dfrac{d\beta}{dt}$

$90\sqrt{2} = 60\sqrt{2}\,\dfrac{d\beta}{dt}$

$\dfrac{d\beta}{dt} = 1.5\ \text{rads}^{-1}$ A1

[7]

Exercise 49

1. $s^2 + 4s + 5t^4 = 0$

$\frac{d}{dt}(s^2) + \frac{d}{dt}(4s) + \frac{d}{dt}(5t^4) = \frac{d}{dt}(0)$ (M1) for valid approach

$2s\frac{ds}{dt} + 4\frac{ds}{dt} + 20t^3 = 0$ (A1) for correct approach

$(2s+4)\frac{ds}{dt} = -20t^3$

$\frac{ds}{dt} = -\frac{20t^3}{2s+4}$

$\frac{ds}{dt} = -\frac{10t^3}{s+2}$ A1

$\frac{d^2s}{dt^2} = -\frac{(s+2)\frac{d}{dt}(10t^3) - (10t^3)\frac{d}{dt}(s+2)}{(s+2)^2}$ (A1) for correct approach

$\frac{d^2s}{dt^2} = -\frac{(s+2)(30t^2) - 10t^3\frac{ds}{dt}}{(s+2)^2}$

$\frac{d^2s}{dt^2} = -\frac{30t^2}{s+2} + \frac{10t^3}{(s+2)^2}\left(-\frac{10t^3}{s+2}\right)$ (M1) for substitution

$\frac{d^2s}{dt^2} = -\frac{30t^2}{s+2} - \frac{100t^6}{(s+2)^3}$ A1

[6]

2. $s=\tan^{-1}t^2$

$\frac{\mathrm{d}s}{\mathrm{d}t}=\left(\frac{1}{1+(t^2)^2}\right)(2t)$ (A1) for correct approach

$\frac{\mathrm{d}s}{\mathrm{d}t}=\frac{2t}{1+t^4}$

$\frac{\mathrm{d}^2s}{\mathrm{d}t^2}=\frac{(1+t^4)(2)-(2t)(4t^3)}{(1+t^4)^2}$ (A1) for correct approach

$\frac{\mathrm{d}^2s}{\mathrm{d}t^2}=\frac{2+2t^4-8t^4}{(1+t^4)^2}$

$\frac{\mathrm{d}^2s}{\mathrm{d}t^2}=\frac{2-6t^4}{(1+t^4)^2}$ A1

$\therefore \left.\frac{\mathrm{d}^2s}{\mathrm{d}t^2}\right|_{t=\sqrt{2}}=\frac{2-6(\sqrt{2})^4}{(1+(\sqrt{2})^4)^2}$ (M1) for substitution

$\left.\frac{\mathrm{d}^2s}{\mathrm{d}t^2}\right|_{t=\sqrt{2}}=-\frac{22}{25}\ \mathrm{ms}^{-2}$ A1

[5]

3. $s=\log_4(3-2e^t)$

$\frac{\mathrm{d}s}{\mathrm{d}t}=\left(\frac{1}{(3-2e^t)\ln 4}\right)(-2e^t)$ A1

$\frac{\mathrm{d}s}{\mathrm{d}t}=-\frac{2e^t}{(3-2e^t)\ln 4}$

$\frac{\mathrm{d}^2s}{\mathrm{d}t^2}=-\frac{1}{\ln 4}\left[\frac{(3-2e^t)(2e^t)-(2e^t)(-2e^t)}{(3-2e^t)^2}\right]$ A1

$\frac{\mathrm{d}^2s}{\mathrm{d}t^2}=-\frac{1}{\ln 4}\left[\frac{6e^t-4e^{2t}+4e^{2t}}{(3-2e^t)^2}\right]$ A1

$\frac{\mathrm{d}^2s}{\mathrm{d}t^2}=-\frac{6e^t}{(3-2e^t)^2\ln 4}$ A1

$\therefore \left.\frac{\mathrm{d}^2s}{\mathrm{d}t^2}\right|_{t=\ln 2}=-\frac{6e^{\ln 2}}{(3-2e^{\ln 2})^2\ln 4}$ M1

$\left.\frac{\mathrm{d}^2s}{\mathrm{d}t^2}\right|_{t=\ln 2}=-\frac{6(2)}{(3-2(2))^2\ln 4}$

$\left.\frac{\mathrm{d}^2s}{\mathrm{d}t^2}\right|_{t=\ln 2}=-\frac{12}{\ln 4}\ \mathrm{ms}^{-2}$ AG

[5]

4. (a) $s = t\arccos t$

$$\frac{ds}{dt} = (1)(\arccos t) + (t)\left(-\frac{1}{\sqrt{1-t^2}}\right)$$ (A1) for correct approach

$$\frac{ds}{dt} = \arccos t - \frac{t}{\sqrt{1-t^2}}$$

$$\frac{ds}{dt} = 0$$ (M1) for setting equation

By considering the graph of $y = \arccos t - \frac{t}{\sqrt{1-t^2}}$,

$t = 0.6521846$.

$\therefore t = 0.652$ A1

[3]

(b) $$\frac{d^2s}{dt^2} = -\frac{1}{\sqrt{1-t^2}} - \frac{(\sqrt{1-t^2})(1) - (t)\left(\frac{1}{2\sqrt{1-t^2}}\right)(-2t)}{(\sqrt{1-t^2})^2}$$ (A1) for correct approach

$$\frac{d^2s}{dt^2} = -\frac{1}{\sqrt{1-t^2}} - \frac{\sqrt{1-t^2} + \frac{t^2}{\sqrt{1-t^2}}}{1-t^2}$$

$$\frac{d^2s}{dt^2} = -\frac{1-t^2}{(1-t^2)^{\frac{3}{2}}} - \frac{1-t^2}{(1-t^2)^{\frac{3}{2}}} - \frac{t^2}{(1-t^2)^{\frac{3}{2}}}$$ (M1) for valid approach

$$\frac{d^2s}{dt^2} = \frac{-2+t^2}{(1-t^2)^{\frac{3}{2}}}$$

$\therefore k = -2$ A1

[3]

Exercise 50

1. (a) The triangles ABP and PCD are similar.

$\therefore \frac{\text{CD}}{x} = \frac{11-x}{4}$ (M1) for valid approach

$\text{CD} = \frac{1}{4}x(11-x)$

$\text{CD} = -\frac{1}{4}x^2 + \frac{11}{4}x$ A1

[2]

(b) (i) $H = \frac{(4)(x)}{2} + \frac{(11-x)\left(-\frac{1}{4}x^2 + \frac{11}{4}x\right)}{2}$ (M1) for valid approach

$H = 2x + (11-x)\left(-\frac{1}{8}x^2 + \frac{11}{8}x\right)$

$H = 2x - \frac{11}{8}x^2 + \frac{121}{8}x + \frac{1}{8}x^3 - \frac{11}{8}x^2$ (A1) for correct approach

$H = \frac{1}{8}x^3 - \frac{11}{4}x^2 + \frac{137}{8}x$ A1

(ii) $\frac{\mathrm{d}H}{\mathrm{d}x} = \frac{1}{8}(3x^2) - \frac{11}{4}(2x) + \frac{137}{8}(1)$ (A1) for correct approach

$\frac{\mathrm{d}H}{\mathrm{d}x} = \frac{3}{8}x^2 - \frac{11}{2}x + \frac{137}{8}$

$\frac{\mathrm{d}H}{\mathrm{d}x} = 0$

$\therefore \frac{3}{8}x^2 - \frac{11}{2}x + \frac{137}{8} = 0$ (M1) for setting equation

$3x^2 - 44x + 137 = 0$

By considering the graph of

$y = 3x^2 - 44x + 137$, $x = 4.4853321$ or

$x = 10.181335$.

By the first derivative test, M1A1

x	$0 < x < 4.4853321$	$x = 4.4853321$	$4.4853321 < x < 10.181335$
$\mathrm{d}H/\mathrm{d}x$	+	0	–
x	$x = 10.181335$	$10.181335 < x < 11$	$x = 11$
$\mathrm{d}H/\mathrm{d}x$	0	+	+

Thus, H attains its local maximum at

$x = 4.4853321$. R1

When $x = 4.4853321$,

$H = \frac{1}{8}(4.4853321)^3 - \frac{11}{4}(4.4853321)^2$

$+\frac{137}{8}(4.4853321)$ (M1) for susbtitution

$H = 32.76585438$

When $x = 11$,

$H = \frac{1}{8}(11)^3 - \frac{11}{4}(11)^2 + \frac{137}{8}(11)$

$H = 22$

Hence, the maximum value of H is 32.8. A1

(iii) $x = 4.49$ A1

[11]

(c) $\tan\theta = \frac{4}{x}$

$\sec^2\theta \frac{d\theta}{dx} = -\frac{4}{x^2}$ (A1) for correct approach

$\frac{d\theta}{dx} = -\frac{4\cos^2\theta}{x^2}$

$\frac{d\theta}{dx} = -\frac{4}{x^2}\left(\frac{x}{\sqrt{4^2 + x^2}}\right)^2$ (M1) for valid approach

$\frac{d\theta}{dx} = -\frac{4}{16 + x^2}$ A1

[3]

(d) $\frac{dH}{dx} = \frac{dH}{d\theta} \cdot \frac{d\theta}{dx}$ (M1) for valid approach

$\frac{3}{8}x^2 - \frac{11}{2}x + \frac{137}{8} = \frac{dH}{d\theta}\left(-\frac{4}{16 + x^2}\right)$ (A1) for substitution

$\frac{3x^2 - 44x + 137}{8} = \frac{dH}{d\theta}\left(-\frac{4}{16 + x^2}\right)$

$\frac{dH}{d\theta} = -\frac{(3x^2 - 44x + 137)(16 + x^2)}{32}$ A1

[3]

2. (a) $K = 2\left(\frac{1}{2}(x-1)(6-x)^2 \sin 60^\circ\right)$ (A1) for substitution

$K = (x-1)(36-12x+x^2)\left(\frac{\sqrt{3}}{2}\right)$

$K = \frac{\sqrt{3}}{2}(-36+48x-13x^2+x^3)$ A1

[2]

(b) (i) $\frac{\mathrm{d}K}{\mathrm{d}x} = \frac{\sqrt{3}}{2}(0+48(1)-13(2x)+3x^2)$ (A1) for correct approach

$\frac{\mathrm{d}K}{\mathrm{d}x} = \frac{\sqrt{3}}{2}(48-26x+3x^2)$ A1

(ii) $\frac{\mathrm{d}K}{\mathrm{d}x} = 0$

$\therefore \frac{\sqrt{3}}{2}(48-26x+3x^2) = 0$ (M1) for setting equation

$3x^2 - 26x + 48 = 0$

$(3x-8)(x-6) = 0$

$x = \frac{8}{3}$ or $x = 6$ (*Rejected*)

By the first derivative test, M1A1

x	$0 < x < \frac{8}{3}$	$x = \frac{8}{3}$	$\frac{8}{3} < x < 6$
$\frac{\mathrm{d}K}{\mathrm{d}x}$	+	0	–

Thus, K attains its maximum at $x = \frac{8}{3}$. R1

The maximum value of K

$= \frac{\sqrt{3}}{2}\left(-36+48\left(\frac{8}{3}\right)-13\left(\frac{8}{3}\right)^2+\left(\frac{8}{3}\right)^3\right)$

$= \frac{250\sqrt{3}}{27}$ A1

(iii) $x = \frac{8}{3}$ A1

[8]

(c) $BC^2 = BD^2 + CD^2 - 2(BD)(CD)\cos B\hat{D}C$ (M1) for cosine rule

$BC^2 = (x-1)^2 + (6-x)^4 - 2(x-1)(6-x)^2 \cos 60^\circ$ (A1) for substitution

$BC^2 = (x-1)^2 + (6-x)^4 - (x-1)(6-x)^2$

$BC = \sqrt{(x-1)^2 + (6-x)^4 - (x-1)(6-x)^2}$

By considering the graph of

$y = \sqrt{(x-1)^2 + (6-x)^4 - (x-1)(6-x)^2}$,

$3.0634008 \le y < 25$.

Thus, the range of values of BC is $3.06 \le BC < 25$. A2

[4]

(d) When K attains its maximum, $x = \frac{8}{3}$.

$BC = \sqrt{\left(\frac{8}{3}-1\right)^2 + \left(6-\frac{8}{3}\right)^4 - \left(\frac{8}{3}-1\right)\left(6-\frac{8}{3}\right)^2}$ (M1) for substitution

$BC = 10.378634$ A1

$BC < 25$

Therefore, BC does not attain its maximum when K attains its maximum.

Thus, the claim is disagreed. A1

[3]

3. (a) $DP^2 + 250^2 = x^2$

$DP = \sqrt{x^2 - 62500}$ (A1) for correct approach

$T = \frac{AP}{2} + \frac{PC}{4}$ (M1) for valid approach

$T = \frac{x}{2} + \frac{500 - \sqrt{x^2 - 62500}}{4}$

$T = \frac{2x + 500 - \sqrt{x^2 - 62500}}{4}$ A1

[3]

(b) (i) $\frac{dT}{dx} = \frac{1}{4}\left(2(1) + 0 - \frac{1}{2}(x^2 - 62500)^{-\frac{1}{2}}(2x - 0)\right)$ (A1) for correct approach

$\frac{dT}{dx} = \frac{1}{4}\left(2 - \frac{x}{\sqrt{x^2 - 62500}}\right)$

$\frac{dT}{dx} = \frac{2\sqrt{x^2 - 62500} - x}{4\sqrt{x^2 - 62500}}$ A1

(ii) $\frac{dT}{dx} = 0$

$\therefore \frac{2\sqrt{x^2 - 62500} - x}{4\sqrt{x^2 - 62500}} = 0$ (M1) for setting equation

$2\sqrt{x^2 - 62500} - x = 0$

$2\sqrt{x^2 - 62500} = x$

$4(x^2 - 62500) = x^2$

$4x^2 - 250000 = x^2$ (A1) for correct approach

$3x^2 = 250000$

$x^2 = \frac{250000}{3}$

$x = \frac{500\sqrt{3}}{3}$ or $x = -\frac{500\sqrt{3}}{3}$ (*Rejected*)

By the first derivative test, M1A1

x	$250 < x < \frac{500\sqrt{3}}{3}$	$x = \frac{500\sqrt{3}}{3}$	$x > \frac{500\sqrt{3}}{3}$
$\frac{dT}{dx}$	$-$	0	$+$

Thus, T attains its minimum at $x = \frac{500\sqrt{3}}{3}$. R1

The minimum value of T

$$= \frac{2\left(\frac{500\sqrt{3}}{3}\right)+500-\sqrt{\left(\frac{500\sqrt{3}}{3}\right)^2-62500}}{4}$$

$= 233.2531755$

$= 233\text{ s}$ A1

(iii) $x = \frac{500\sqrt{3}}{3}$ A1

[9]

(c) $L = AP + PC$

$L = x + 500 - \sqrt{x^2 - 62500}$ A1

$\frac{dL}{dx} = 1 + 0 - \frac{1}{2}(x^2 - 62500)^{-\frac{1}{2}}(2x - 0)$ A1

$\frac{dL}{dx} = 1 - \frac{x}{\sqrt{x^2 - 62500}}$ AG

[2]

(d) When T attains its minimum, $x = \frac{500\sqrt{3}}{3}$.

By considering the graph of

$y = x + 500 - \sqrt{x^2 - 62500}$, the graph of L is

concave upward. (M1) for valid approach

$\therefore \frac{d^2L}{dx^2} > 0$ for $x \geq 250$

$\frac{d}{dx}\left(\frac{dL}{dx}\right) > 0$ for $x \geq 250$ R1

Therefore, $\frac{dL}{dx}$ is increasing when T attains its minimum.

Thus, the claim is agreed. A1

[3]

4. (a) $g(x)=e^{-x^2}$

$g'(x)=(e^{-x^2})(-2x)$ (A1) for correct approach

$g'(x)=-2xe^{-x^2}$

$g''(x)=(-2)(e^{-x^2})+(-2x)(e^{-x^2})(-2x)$ (A1) for correct approach

$g''(x)=-2e^{-x^2}+4x^2e^{-x^2}$

$g''(x)=2e^{-x^2}(2x^2-1)$ A1

[3]

(b) $0<x<\frac{1}{\sqrt{2}}$

$0<x^2<\frac{1}{2}$

$0<2x^2<1$

$-1<2x^2-1<0$ A1

$e^{-x^2}>0$ for all values of x. A1

$\therefore g''(x)<0$ for $0<x<\frac{1}{\sqrt{2}}$.

Thus, the graph of $g(x)$ is concave downward for

$0<x<\frac{1}{\sqrt{2}}$. AG

[2]

(c) (i) $\text{OA}=e^{-0^2}-0$

$\text{OA}=1$

$\text{BB}'=e^{-h^2}-0$

$\text{BB}'=e^{-h^2}$ (A1) for correct value

$\text{CC}'=e^{-\left(\frac{1}{\sqrt{2}}\right)^2}-0$

$\text{CC}'=e^{-\frac{1}{2}}$ (A1) for correct value

$T=\frac{(1+e^{-h^2})(h)}{2}+\frac{(e^{-h^2}+e^{-\frac{1}{2}})\left(\frac{1}{\sqrt{2}}-h\right)}{2}$ (A1) for correct approach

$T=\frac{h+he^{-h^2}}{2}$

$+\frac{\frac{1}{\sqrt{2}}e^{-h^2}+\frac{1}{\sqrt{2}}e^{-\frac{1}{2}}-he^{-h^2}-he^{-\frac{1}{2}}}{2}$

$$T=\frac{2h+\sqrt{2}e^{-h^2}+\sqrt{2}e^{-\frac{1}{2}}-2he^{-\frac{1}{2}}}{4}$$ A1

(ii) $\frac{\mathrm{d}T}{\mathrm{d}h}=\frac{1}{4}\left(2(1)+\sqrt{2}e^{-h^2}(-2h)+0-2e^{-\frac{1}{2}}(1)\right)$ (A1) for correct approach

$\frac{\mathrm{d}T}{\mathrm{d}h}=\frac{1}{4}\left(2-2\sqrt{2}he^{-h^2}-2e^{-\frac{1}{2}}\right)$

$\frac{\mathrm{d}T}{\mathrm{d}h}=\frac{1}{2}\left(1-\sqrt{2}he^{-h^2}-e^{-\frac{1}{2}}\right)$ A1

(iii) $\frac{\mathrm{d}T}{\mathrm{d}h}=0$

$\therefore\frac{1}{2}\left(1-\sqrt{2}he^{-h^2}-e^{-\frac{1}{2}}\right)=0$ (M1) for setting equation

$1-\sqrt{2}he^{-h^2}-e^{-\frac{1}{2}}=0$

By considering the graph of

$y=1-\sqrt{2}he^{-h^2}-e^{-\frac{1}{2}}$, $h=0.3054287$.

By the first derivative test, M1A1

h	$h=0$	$0<h<0.3054287$	$h=0.3054287$	$0.3054287<h<\frac{1}{\sqrt{2}}$	$h=\frac{1}{\sqrt{2}}$
$\frac{\mathrm{d}T}{\mathrm{d}h}$	+	+	0	–	–

Thus, T attains its maximum at

$h=0.3054287$. R1

The maximum value of T

$$=\frac{2(0.3054287)+\sqrt{2}e^{-0.3054287^2}+\sqrt{2}e^{-\frac{1}{2}}-2(0.3054287)e^{-\frac{1}{2}}}{4}$$

$=0.5965926$

$=0.597$ A1

(iv) $h=0.3054287$ A1

[12]

(d) $T \geq \dfrac{2(0)+\sqrt{2}e^{-0^2}+\sqrt{2}e^{-\frac{1}{2}}-2(0)e^{-\frac{1}{2}}}{4}$ M1

$T \geq \dfrac{\sqrt{2}+\sqrt{2}e^{-\frac{1}{2}}}{4}$ A1

$T \geq \dfrac{\sqrt{2}\left(1+e^{-\frac{1}{2}}\right)}{4}$

$T \geq \dfrac{\sqrt{2}}{4}\left(1+\dfrac{1}{\sqrt{e}}\right)$ AG

[2]

Chapter 12 Solution

Exercise 51

1. $\int_0^k e^{4x+3}\mathrm{d}x = \frac{1}{4}e^3(e^{28}-1)$

Let $u = e^{4x+3}$. (M1) for substitution

$\frac{\mathrm{d}u}{\mathrm{d}x} = 4e^{4x+3} \Rightarrow \frac{1}{4}\mathrm{d}u = e^{4x+3}\mathrm{d}x$

$x = k \Rightarrow u = e^{4k+3}$

$x = 0 \Rightarrow u = e^{4(0)+3} = e^3$

$\therefore \int_{e^3}^{e^{4k+3}} \frac{1}{4}\mathrm{d}u = \frac{1}{4}e^3(e^{28}-1)$ (A2) for correct working

$\left[\frac{1}{4}u\right]_{e^3}^{e^{4k+3}} = \frac{1}{4}e^3(e^{28}-1)$ A1

$\frac{1}{4}e^{4k+3} - \frac{1}{4}e^3 = \frac{1}{4}e^{31} - \frac{1}{4}e^3$

$4k+3 = 31$

$4k = 28$

$k = 7$ A1

[5]

2. $\int_{-\frac{\pi}{3}}^{0} \frac{\sin^3 x}{\cos x} dx = \frac{k}{8} - \ln 2$

$\int_{-\frac{\pi}{3}}^{0} \frac{\sin^2 x \cdot \sin x}{\cos x} dx = \frac{k}{8} - \ln 2$

Let $u = \cos x$. (M1) for substitution

$\frac{du}{dx} = -\sin x \Rightarrow (-1)du = \sin x dx$

$1 - u^2 = \sin^2 x$

$x = 0 \Rightarrow u = \cos 0 = 1$

$x = -\frac{\pi}{3} \Rightarrow u = \cos\left(-\frac{\pi}{3}\right) = \frac{1}{2}$

$\therefore \int_{\frac{1}{2}}^{1} \frac{1-u^2}{u}(-1)du = \frac{k}{8} - \ln 2$ (A2) for correct working

$\therefore \int_{\frac{1}{2}}^{1} \left(u - \frac{1}{u}\right) du = \frac{k}{8} - \ln 2$

$\left[\frac{1}{2}u^2 - \ln|u|\right]_{\frac{1}{2}}^{1} = \frac{k}{8} - \ln 2$ A1

$\left(\frac{1}{2}(1)^2 - \ln 1\right) - \left(\frac{1}{2}\left(\frac{1}{2}\right)^2 - \ln \frac{1}{2}\right) = \frac{k}{8} - \ln 2$

$\frac{1}{2} - \left(\frac{1}{8} + \ln 2\right) = \frac{k}{8} - \ln 2$ A1

$\frac{3}{8} - \ln 2 = \frac{k}{8} - \ln 2$

$k = 3$ A1

[6]

3. $\int_{e}^{e^c} \frac{1}{x(\ln x)^2}\,dx = \frac{8}{9}$

Let $u = \ln x$. (M1) for substitution

$\frac{du}{dx} = \frac{1}{x} \Rightarrow du = \frac{1}{x}dx$

$x = e^c \Rightarrow u = \ln e^c = c$

$x = e \Rightarrow u = \ln e = 1$

$\therefore \int_{1}^{c} \frac{1}{u^2}\,du = \frac{8}{9}$ (A2) for correct working

$\left[-\frac{1}{u}\right]_1^c = \frac{8}{9}$ A1

$-\frac{1}{c} - \left(-\frac{1}{1}\right) = \frac{8}{9}$

$-\frac{1}{c} = -\frac{1}{9}$

$c = 9$ A1

[5]

4. $\int_k^4 \frac{(\sqrt{x}+x)(1+2\sqrt{x})}{2\sqrt{x}}\,\mathrm{d}x = 16$

$\int_k^4 (\sqrt{x}+x)\left(\frac{1}{2\sqrt{x}}+1\right)\mathrm{d}x = 16$

Let $u=\sqrt{x}+x$. (M1) for substitution

$\frac{\mathrm{d}u}{\mathrm{d}x}=\frac{1}{2\sqrt{x}}+1 \Rightarrow \mathrm{d}u=\left(\frac{1}{2\sqrt{x}}+1\right)\mathrm{d}x$

$x=4 \Rightarrow u=\sqrt{4}+4=6$

$x=k \Rightarrow u=\sqrt{k}+k$

$\therefore \int_{\sqrt{k}+k}^{6} u\,\mathrm{d}u = 16$ (A2) for correct working

$\left[\frac{1}{2}u^2\right]_{\sqrt{k}+k}^{6} = 16$ A1

$\frac{1}{2}(6)^2-\frac{1}{2}(\sqrt{k}+k)^2=16$ A1

$36-(\sqrt{k}+k)^2=32$

$(\sqrt{k}+k)^2=4$

$\sqrt{k}+k=-2$ (*Rejected*) or $\sqrt{k}+k=2$ A1

$(\sqrt{k})^2+\sqrt{k}-2=0$

$(\sqrt{k}+2)(\sqrt{k}-1)=0$

$\sqrt{k}=-2$ (*Rejected*) or $\sqrt{k}=1$

$k=1$ A1

[7]

Exercise 52

1. $\int_1^4 \ln\frac{x}{4}\,dx = \left[(x)\left(\ln\frac{x}{4}\right)\right]_1^4 - \int_1^4 x\,d\left(\ln\frac{x}{4}\right)$ (A2) for correct working

$\int_1^4 \ln\frac{x}{4}\,dx = \left[x\ln\frac{x}{4}\right]_1^4 - \int_1^4 x\cdot\frac{1}{x}\,dx$

$\int_1^4 \ln\frac{x}{4}\,dx = \left[x\ln\frac{x}{4}\right]_1^4 - \int_1^4 dx$ (M1) for valid approach

$\int_1^4 \ln\frac{x}{4}\,dx = \left[x\ln\frac{x}{4}\right]_1^4 - [x]_1^4$ A1

$\int_1^4 \ln\frac{x}{4}\,dx = \left[x\ln\frac{x}{4} - x\right]_1^4$

$\int_1^4 \ln\frac{x}{4}\,dx = (4\ln 1 - 4) - \left(\ln\frac{1}{4} - 1\right)$

$\int_1^4 \ln\frac{x}{4}\,dx = -4 - \ln\frac{1}{4} + 1$

$\int_1^4 \ln\frac{x}{4}\,dx = -3 - \ln\frac{1}{4}$ A1

[5]

2. $\int_1^4 x\ln x\,dx = k\ln 4 - \frac{15}{4}$

Let $\theta = x^2$. (M1) for valid approach

$\frac{d\theta}{dx} = 2x \Rightarrow \frac{1}{2}d\theta = 2x\,dx$

$\therefore \int_1^4 \frac{1}{2}\ln x\,d(x^2) = k\ln 4 - \frac{15}{4}$

$\left[\frac{1}{2}(\ln x)(x^2)\right]_1^4 - \frac{1}{2}\int_1^4 x^2 d(\ln x) = k\ln 4 - \frac{15}{4}$ (A2) for correct working

$\left[\frac{1}{2}x^2\ln x\right]_1^4 - \frac{1}{2}\int_1^4 x^2 \cdot \frac{1}{x}dx = k\ln 4 - \frac{15}{4}$

$\left[\frac{1}{2}x^2\ln x\right]_1^4 - \frac{1}{2}\int_1^4 x\,dx = k\ln 4 - \frac{15}{4}$

$\left[\frac{1}{2}x^2\ln x\right]_1^4 - \frac{1}{2}\left[\frac{1}{2}x^2\right]_1^4 = k\ln 4 - \frac{15}{4}$ A1

$\left[\frac{1}{2}x^2\ln x - \frac{1}{4}x^2\right]_1^4 = k\ln 4 - \frac{15}{4}$

$\left(\frac{1}{2}(4)^2\ln 4 - \frac{1}{4}(4)^2\right) - \left(\frac{1}{2}(1)^2\ln 1 - \frac{1}{4}(1)^2\right)$

$= k\ln 4 - \frac{15}{4}$

$(8\ln 4 - 4) - \left(-\frac{1}{4}\right) = k\ln 4 - \frac{15}{4}$ A1

$8\ln 4 - \frac{15}{4} = k\ln 4 - \frac{15}{4}$

$k = 8$ A1

[6]

3. Let $\theta = \cos 4x$. (M1) for valid approach

$$\frac{d\theta}{dx} = -4\sin 4x \Rightarrow -\frac{1}{4}d\theta = \sin 4x dx$$

$$\therefore \int x^2 \sin 4x dx = \int -\frac{1}{4}x^2 d(\cos 4x)$$

$$\int x^2 \sin 4x dx = \left(-\frac{1}{4}x^2\right)(\cos 4x) - \int \cos 4x d\left(-\frac{1}{4}x^2\right)$$ (A2) for correct working

$$\int x^2 \sin 4x dx = -\frac{1}{4}x^2 \cos 4x + \int \frac{1}{2}x \cos 4x dx$$

Let $\alpha = \sin 4x$. (M1) for valid approach

$$\frac{d\alpha}{dx} = 4\cos 4x \Rightarrow \frac{1}{4}d\alpha = \cos 4x dx$$

$$\therefore \int x^2 \sin 4x dx = -\frac{1}{4}x^2 \cos 4x + \int \frac{1}{8}x d(\sin 4x)$$

$$\int x^2 \sin 4x dx = -\frac{1}{4}x^2 \cos 4x$$

$$+\left(\frac{1}{8}x\right)(\sin 4x) - \int \sin 4x d\left(\frac{1}{8}x\right)$$ (A1) for correct working

$$\int x^2 \sin 4x dx = -\frac{1}{4}x^2 \cos 4x + \frac{1}{8}x \sin 4x - \int \frac{1}{8}\sin 4x dx$$

$$\int x^2 \sin 4x dx = -\frac{1}{4}x^2 \cos 4x + \frac{1}{8}x \sin 4x + \frac{1}{32}\cos 4x + C$$ A1

[6]

4. Let $\theta = \sin x$. (M1) for valid approach

$$\frac{d\theta}{dx} = \cos x \Rightarrow d\theta = \cos x dx$$

$$\therefore \int e^{3x} \cos x dx = \int e^{3x} d(\sin x)$$

$$\int e^{3x} \cos x dx = e^{3x} \sin x - \int \sin x d(e^{3x})$$ (A2) for correct working

$$\int e^{3x} \cos x dx = e^{3x} \sin x - \int 3 \sin x e^{3x} dx$$

Let $\alpha = \cos x$. (M1) for valid approach

$$\frac{d\alpha}{dx} = -\sin x \Rightarrow d\alpha = -\sin x dx$$

$$\therefore \int e^{3x} \cos x dx = e^{3x} \sin x + \int 3e^{3x} d(\cos x)$$

$$\int e^{3x} \cos x dx = e^{3x} \sin x + 3e^{3x} \cos x - \int \cos x d(3e^{3x})$$ (A1) for correct working

$$\int e^{3x} \cos x dx = e^{3x} \sin x + 3e^{3x} \cos x - \int 9e^{3x} \cos x dx$$

$$\therefore 10 \int e^{3x} \cos x dx = e^{3x} \sin x + 3e^{3x} \cos x + C$$ A1

$$\int_0^{\pi} e^{3x} \cos x dx = \frac{1}{10}\left[e^{3x} \sin x + 3e^{3x} \cos x\right]_0^{\pi}$$

$$\int_0^{\pi} e^{3x} \cos x dx = \frac{1}{10}\begin{bmatrix}(e^{3\pi} \sin \pi + 3e^{3\pi} \cos \pi) \\ -(e^{3(0)} \sin 0 + 3e^{3(0)} \cos 0)\end{bmatrix}$$ A1

$$\int_0^{\pi} e^{3x} \cos x dx = \frac{1}{10}(-3e^{3\pi} - 3)$$

$$\int_0^{\pi} e^{3x} \cos x dx = -\frac{3}{10}(e^{3\pi} + 1)$$ AG

[7]

Exercise 53

1. (a) $f(x)=\dfrac{1}{(x+2)(x-4)}$

Let $\dfrac{1}{(x+2)(x-4)}\equiv\dfrac{A}{x+2}+\dfrac{B}{x-4}$, where A and B are constants.

$$\frac{1}{(x+2)(x-4)}\equiv\frac{A(x-4)}{(x+2)(x-4)}+\frac{B(x+2)}{(x+2)(x-4)}$$ M1

$$\frac{1}{(x+2)(x-4)}\equiv\frac{Ax-4A+Bx+2B}{(x+2)(x-4)}$$

$1\equiv(A+B)x+(-4A+2B)$ A1

$0=A+B$

$B=-A$

$1=-4A+2B$

$\therefore 1=-4A+2(-A)$ A1

$1=-6A$

$A=-\dfrac{1}{6}$

$\therefore B=-\left(-\dfrac{1}{6}\right)$

$B=\dfrac{1}{6}$

$$\therefore \frac{1}{(x+2)(x-4)}\equiv-\frac{1}{6(x+2)}+\frac{1}{6(x-4)}$$ A1

[4]

(b) $\displaystyle\int f(x)\mathrm{d}x=\int\left(-\frac{1}{6(x+2)}+\frac{1}{6(x-4)}\right)\mathrm{d}x$

$\displaystyle\int f(x)\mathrm{d}x=-\frac{1}{6}\ln(x+2)+\frac{1}{6}\ln(x-4)+C$ A1

[1]

2. (a) $f(x)=\dfrac{3}{x(x-5)}$

Let $\dfrac{3}{x(x-5)}\equiv\dfrac{A}{x}+\dfrac{B}{x-5}$, where A and B are constants.

$\dfrac{3}{x(x-5)}\equiv\dfrac{A(x-5)}{x(x-5)}+\dfrac{Bx}{x(x-5)}$ M1

$\dfrac{3}{x(x-5)}\equiv\dfrac{Ax-5A+Bx}{x(x-5)}$

$3\equiv(A+B)x-5A$ A1

$0=A+B$

$B=-A$ A1

$3=-5A$

$A=-\dfrac{3}{5}$

$\therefore B=-\left(-\dfrac{3}{5}\right)$

$B=\dfrac{3}{5}$

$\therefore\dfrac{3}{x(x-5)}\equiv-\dfrac{3}{5x}+\dfrac{3}{5(x-5)}$ A1

[4]

(b) $\int_6^{11}f(t)\mathrm{d}t=\int_6^{11}\left(-\dfrac{3}{5t}+\dfrac{3}{5(t-5)}\right)\mathrm{d}t$

$\int_6^{11}f(t)\mathrm{d}t=\left[-\dfrac{3}{5}\ln t+\dfrac{3}{5}\ln(t-5)\right]_6^{11}$ A1

$\int_6^{11}f(t)\mathrm{d}t=\left(-\dfrac{3}{5}\ln 11+\dfrac{3}{5}\ln(11-5)\right)$

$-\left(-\dfrac{3}{5}\ln 6+\dfrac{3}{5}\ln(6-5)\right)$

$\int_6^{11}f(t)\mathrm{d}t=-\dfrac{3}{5}\ln 11+\dfrac{3}{5}\ln 6+\dfrac{3}{5}\ln 6-\dfrac{3}{5}\ln 1$

$\int_6^{11}f(t)\mathrm{d}t=-\dfrac{3}{5}\ln 11+\dfrac{6}{5}\ln 6$ A1

[2]

3. (a) $x=-2,\ x=-1$ A2

[2]

(b) $f(x)=\dfrac{x+4}{(x+2)(x+1)}$

Let $\dfrac{x+4}{(x+2)(x+1)}\equiv\dfrac{A}{x+2}+\dfrac{B}{x+1}$, where A and B are constants.

$\dfrac{x+4}{(x+2)(x+1)}\equiv\dfrac{A(x+1)}{(x+2)(x+1)}+\dfrac{B(x+2)}{(x+2)(x+1)}$ M1

$\dfrac{x+4}{(x+2)(x+1)}\equiv\dfrac{Ax+A+Bx+2B}{(x+2)(x+1)}$

$x+4\equiv(A+B)x+(A+2B)$ A1

$1=A+B$

$B=1-A$

$4=A+2B$

$\therefore 4=A+2(1-A)$ A1

$2=-A$

$A=-2$

$\therefore B=1-(-2)$

$B=3$

$\therefore \dfrac{x+4}{(x+2)(x+1)}\equiv-\dfrac{2}{x+2}+\dfrac{3}{x+1}$ A1

[4]

(c) $\int_4^5 f(x)\mathrm{d}x=\int_4^5\left(-\dfrac{2}{x+2}+\dfrac{3}{x+1}\right)\mathrm{d}x$

$\int_4^5 f(x)\mathrm{d}x=\left[-2\ln(x+2)+3\ln(x+1)\right]_4^5$ A1

$\int_4^5 f(x)\mathrm{d}x=(-2\ln(5+2)+3\ln(5+1))$

$-(-2\ln(4+2)+3\ln(4+1))$

$\int_4^5 f(x)\mathrm{d}x=-2\ln7+3\ln6+2\ln6-3\ln5$

$\int_4^5 f(x)\mathrm{d}x=-2\ln7+5\ln6-3\ln5$ A1

[2]

4. (a) 0 A1

[1]

(b) $f(x)=\dfrac{1-2x}{(x+5)(x-3)}$

Let $\dfrac{1-2x}{(x+5)(x-3)}\equiv\dfrac{A}{x+5}+\dfrac{B}{x-3}$, where A and B are constants.

$$\frac{1-2x}{(x+5)(x-3)}\equiv\frac{A(x-3)}{(x+5)(x-3)}+\frac{B(x+5)}{(x+5)(x-3)}$$ M1

$$\frac{1-2x}{(x+5)(x-3)}\equiv\frac{Ax-3A+Bx+5B}{(x+5)(x-3)}$$

$1-2x\equiv(A+B)x+(-3A+5B)$ A1

$-2=A+B$

$B=-2-A$

$1=-3A+5B$

$\therefore 1=-3A+5(-2-A)$ A1

$1=-3A-10-5A$

$11=-8A$

$A=-\dfrac{11}{8}$

$\therefore B=-2-\left(-\dfrac{11}{8}\right)$

$B=-\dfrac{5}{8}$

$$\therefore \frac{1-2x}{(x+5)(x-3)}\equiv-\frac{11}{8(x+5)}-\frac{5}{8(x-3)}$$ A1

[4]

(c) $$\int_4^{12} f(x)\mathrm{d}x=\int_4^{12}\left(-\frac{11}{8(x+5)}-\frac{5}{8(x-3)}\right)\mathrm{d}x$$

$$\int_4^{12} f(x)\mathrm{d}x=\left[-\frac{11}{8}\ln(x+5)-\frac{5}{8}\ln(x-3)\right]_4^{12}$$ A1

$$\int_4^{12} f(x)\mathrm{d}x=\left(-\frac{11}{8}\ln(12+5)-\frac{5}{8}\ln(12-3)\right)-\left(-\frac{11}{8}\ln(4+5)-\frac{5}{8}\ln(4-3)\right)$$

$$\int_4^{12} f(x)\mathrm{d}x=-\frac{11}{8}\ln 17-\frac{5}{8}\ln 9+\frac{11}{8}\ln 9+\frac{5}{8}\ln 1$$

$$\int_4^{12} f(x)\mathrm{d}x=-\frac{11}{8}\ln 17+\frac{3}{4}\ln 9$$ A1

[2]

Exercise 54

1. $x = 10\sec\theta$

$\frac{dx}{d\theta} = 10\sec\theta\tan\theta \Rightarrow dx = 10\sec\theta\tan\theta d\theta$ M1

$\sec\theta = \frac{x}{10}$

$\theta = \text{arcsec}\frac{x}{10}$ A1

$\therefore \int \frac{1}{x\sqrt{x^2 - 100}}dx = \int \frac{10\sec\theta\tan\theta d\theta}{10\sec\theta\sqrt{100\sec^2\theta - 100}}$ A1

$\int \frac{1}{x\sqrt{x^2 - 100}}dx = \int \frac{\tan\theta}{10\tan\theta}d\theta$ A1

$\int \frac{1}{x\sqrt{x^2 - 100}}dx = \int \frac{1}{10}d\theta$

$\int \frac{1}{x\sqrt{x^2 - 100}}dx = \frac{1}{10}\theta + C$ A1

$\int \frac{1}{x\sqrt{x^2 - 100}}dx = \frac{1}{10}\text{arcsec}\frac{x}{10} + C$ AG

[5]

2. $\int_{\frac{1}{2}}^{\frac{\sqrt{2}}{2}} \frac{1}{\sqrt{1-x^2}}\,dx = \frac{\pi}{k}$

Let $x = \sin\theta$.

$\frac{dx}{d\theta} = \cos\theta \Rightarrow dx = \cos\theta\, d\theta$ (M1) for substitution

$x = \frac{\sqrt{2}}{2} \Rightarrow \theta = \arcsin\frac{\sqrt{2}}{2} = \frac{\pi}{4}$

$x = \frac{1}{2} \Rightarrow \theta = \arcsin\frac{1}{2} = \frac{\pi}{6}$

$\therefore \int_{\frac{\pi}{6}}^{\frac{\pi}{4}} \frac{1}{\sqrt{1-\sin^2\theta}} \cdot \cos\theta\, d\theta = \frac{\pi}{k}$ (A2) for correct working

$\therefore \int_{\frac{\pi}{6}}^{\frac{\pi}{4}} \frac{\cos\theta}{\cos\theta}\, d\theta = \frac{\pi}{k}$ A1

$[\theta]_{\frac{\pi}{6}}^{\frac{\pi}{4}} = \frac{\pi}{k}$ A1

$\frac{\pi}{4} - \frac{\pi}{6} = \frac{\pi}{k}$

$\frac{\pi}{12} = \frac{\pi}{k}$

$k = 12$ A1

[6]

3. (a)

$-x^2-2x+8=-(x^2+2x-8)$

$-x^2-2x+8=-(x^2+2x+1-9)$ (M1) for valid approach

$-x^2-2x+8=9-(x^2+2x+1)$

$-x^2-2x+8=9-(x+1)^2$ A1

[2]

(b)

$$\int_{-\frac{3\sqrt{2}}{2}-1}^{\frac{3\sqrt{2}}{2}-1}\frac{1}{\sqrt{-x^2-2x+8}}\,\mathrm{d}x=\int_{-\frac{3\sqrt{2}}{2}-1}^{\frac{3\sqrt{2}}{2}-1}\frac{1}{\sqrt{9-(x+1)^2}}\,\mathrm{d}x$$

Let $x+1=3\sin\theta$. A1

$x=3\sin\theta-1$

$\frac{\mathrm{d}x}{\mathrm{d}\theta}=3\cos\theta\Rightarrow \mathrm{d}x=3\cos\theta\,\mathrm{d}\theta$ M1

$x=\frac{3\sqrt{2}}{2}-1\Rightarrow\theta=\sin^{-1}\frac{\sqrt{2}}{2}=\frac{\pi}{4}$

$x=-\frac{3\sqrt{2}}{2}-1\Rightarrow\theta=\sin^{-1}\left(-\frac{\sqrt{2}}{2}\right)=-\frac{\pi}{4}$

$\therefore\int_{-\frac{3\sqrt{2}}{2}-1}^{\frac{3\sqrt{2}}{2}-1}\frac{1}{\sqrt{-x^2-2x+8}}\,\mathrm{d}x=\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}}\frac{3\cos\theta}{\sqrt{9-9\sin^2\theta}}\,\mathrm{d}\theta$ A1

$\int_{-\frac{3\sqrt{2}}{2}-1}^{\frac{3\sqrt{2}}{2}-1}\frac{1}{\sqrt{-x^2-2x+8}}\,\mathrm{d}x=\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}}\frac{3\cos\theta}{3\cos\theta}\,\mathrm{d}\theta$ A1

$\int_{-\frac{3\sqrt{2}}{2}-1}^{\frac{3\sqrt{2}}{2}-1}\frac{1}{\sqrt{-x^2-2x+8}}\,\mathrm{d}x=\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}}\mathrm{d}\theta$

$\int_{-\frac{3\sqrt{2}}{2}-1}^{\frac{3\sqrt{2}}{2}-1}\frac{1}{\sqrt{-x^2-2x+8}}\,\mathrm{d}x=[\theta]_{-\frac{\pi}{4}}^{\frac{\pi}{4}}$ A1

$\int_{-\frac{3\sqrt{2}}{2}-1}^{\frac{3\sqrt{2}}{2}-1}\frac{1}{\sqrt{-x^2-2x+8}}\,\mathrm{d}x=\frac{\pi}{4}-\left(-\frac{\pi}{4}\right)$

$\int_{-\frac{3\sqrt{2}}{2}-1}^{\frac{3\sqrt{2}}{2}-1}\frac{1}{\sqrt{-x^2-2x+8}}\,\mathrm{d}x=\frac{\pi}{2}$ AG

[5]

4. (a) $x^2+8x+20=x^2+8x+16+4$ (M1) for valid approach

$x^2+8x+20=(x+4)^2+4$ A1

[2]

(b) $\int_{-4}^{-2}\frac{1}{(x^2+8x+20)^2}\,dx=\int_{-4}^{-2}\frac{1}{((x+4)^2+4)^2}\,dx$

Let $x+4=2\tan\theta$. A1

$x=2\tan\theta-4$

$\frac{dx}{d\theta}=2\sec^2\theta\Rightarrow dx=2\sec^2\theta d\theta$ M1

$x=-2\Rightarrow\theta=\arctan 1=\frac{\pi}{4}$

$x=-4\Rightarrow\theta=\arctan 0=0$

$\therefore\int_{-4}^{-2}\frac{1}{(x^2+8x+20)^2}\,dx=\int_{0}^{\frac{\pi}{4}}\frac{2\sec^2\theta}{(4\tan^2\theta+4)^2}\,d\theta$ A1

$\int_{-4}^{-2}\frac{1}{(x^2+8x+20)^2}\,dx=\int_{0}^{\frac{\pi}{4}}\frac{2\sec^2\theta}{16\sec^4\theta}\,d\theta$ A1

$\int_{-4}^{-2}\frac{1}{(x^2+8x+20)^2}\,dx=\int_{0}^{\frac{\pi}{4}}\frac{1}{8}\cos^2\theta d\theta$

$\int_{-4}^{-2}\frac{1}{(x^2+8x+20)^2}\,dx=\int_{0}^{\frac{\pi}{4}}\frac{1}{8}\left(\frac{1+\cos 2\theta}{2}\right)d\theta$ A1

$\int_{-4}^{-2}\frac{1}{(x^2+8x+20)^2}\,dx=\frac{1}{16}\int_{0}^{\frac{\pi}{4}}(1+\cos 2\theta)d\theta$

$\int_{-4}^{-2}\frac{1}{(x^2+8x+20)^2}\,dx=\frac{1}{16}\left[\theta+\frac{1}{2}\sin 2\theta\right]_0^{\frac{\pi}{4}}$ A1

$\int_{-4}^{-2}\frac{1}{(x^2+8x+20)^2}\,dx$

$=\frac{1}{16}\left(\frac{\pi}{4}+\frac{1}{2}\sin 2\left(\frac{\pi}{4}\right)\right)-\frac{1}{16}\left(0+\frac{1}{2}\sin 2(0)\right)$

$\int_{-4}^{-2}\frac{1}{(x^2+8x+20)^2}\,dx=\frac{\pi}{64}+\frac{1}{32}$ AG

[6]

Chapter 13 Solution

Exercise 55

1. The area of R

$= \int_{\frac{1}{6}}^{\frac{1}{2}} f(x)\mathrm{d}x$ (A1) for correct approach

$= \int_{\frac{1}{6}}^{\frac{1}{2}} \pi\operatorname{cosec}^2 \pi x \mathrm{d}x$

Let $u = \pi x$. (M1) for substitution

$\frac{\mathrm{d}u}{\mathrm{d}x} = \pi \Rightarrow \mathrm{d}u = \pi \mathrm{d}x$

$x = \frac{1}{2} \Rightarrow u = \pi\left(\frac{1}{2}\right) = \frac{\pi}{2}$

$x = \frac{1}{6} \Rightarrow u = \pi\left(\frac{1}{6}\right) = \frac{\pi}{6}$

$\therefore \int_{\frac{1}{6}}^{\frac{1}{2}} f(x)\mathrm{d}x = \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \operatorname{cosec}^2 u \mathrm{d}u$ (A2) for correct working

$\int_{\frac{1}{6}}^{\frac{1}{2}} f(x)\mathrm{d}x = \left[-\cot u\right]_{\frac{\pi}{6}}^{\frac{\pi}{2}}$ A1

$\int_{\frac{1}{6}}^{\frac{1}{2}} f(x)\mathrm{d}x = -\cot\frac{\pi}{2} - \left(-\cot\frac{\pi}{6}\right)$

$\int_{\frac{1}{6}}^{\frac{1}{2}} f(x)\mathrm{d}x = 0 - (-\sqrt{3})$

$\int_{\frac{1}{6}}^{\frac{1}{2}} f(x)\mathrm{d}x = \sqrt{3}$ A1

[6]

2. The area of R

$= \int_1^{e^4} f(x)\mathrm{d}x$ (A1) for correct approach

$= \int_1^{e^4} \ln x\mathrm{d}x$

$= \left[(\ln x)(x)\right]_1^{e^4} - \int_1^{e^4} x\mathrm{d}(\ln x)$ (A2) for correct working

$= \left[x\ln x\right]_1^{e^4} - \int_1^{e^4} x\cdot\frac{1}{x}\mathrm{d}x$

$= \left[x\ln x\right]_1^{e^4} - \int_1^{e^4} 1\mathrm{d}x$ (M1) for valid approach

$= \left[x\ln x\right]_1^{e^4} - \left[x\right]_1^{e^4}$ A1

$= e^4\ln e^4 - 1\ln 1 - (e^4 - 1)$

$= 4e^4 - e^4 + 1$

$= 3e^4 + 1$ A1

[6]

3. $\int_{\sqrt{\frac{\pi}{6}}}^{\frac{\sqrt{\pi}}{k}} f(x)\mathrm{d}x = 2-\sqrt{2}$ (M1) for setting equation

$\int_{\sqrt{\frac{\pi}{6}}}^{\frac{\sqrt{\pi}}{k}} 2x\cot x^2 \operatorname{cosec} x^2 \mathrm{d}x = 2-\sqrt{2}$

Let $u = x^2$. (M1) for substitution

$\frac{\mathrm{d}u}{\mathrm{d}x} = 2x \Rightarrow \mathrm{d}u = 2x\mathrm{d}x$

$x = \frac{\sqrt{\pi}}{k} \Rightarrow u = \left(\frac{\sqrt{\pi}}{k}\right)^2 = \frac{\pi}{k^2}$

$x = \sqrt{\frac{\pi}{6}} \Rightarrow u = \left(\sqrt{\frac{\pi}{6}}\right)^2 = \frac{\pi}{6}$

$\therefore \int_{\frac{\pi}{6}}^{\frac{\pi}{k^2}} \cot u \operatorname{cosec} u \mathrm{d}u = 2-\sqrt{2}$ (A2) for correct working

$\left[-\operatorname{cosec} u\right]_{\frac{\pi}{6}}^{\frac{\pi}{k^2}} = 2-\sqrt{2}$ A1

$-\operatorname{cosec}\frac{\pi}{k^2} - \left(-\operatorname{cosec}\frac{\pi}{6}\right) = 2-\sqrt{2}$

$-\operatorname{cosec}\frac{\pi}{k^2} - (-2) = 2-\sqrt{2}$

$\operatorname{cosec}\frac{\pi}{k^2} = \sqrt{2}$

$\therefore \sin\frac{\pi}{k^2} = \frac{1}{\sqrt{2}}$ (A1) for correct approach

$\frac{\pi}{k^2} = \frac{\pi}{4}$

$k^2 = 4$

$k = 2$ or $k = -2$ (*Rejected*) A1

[7]

4. $\int_2^k f(x)\mathrm{d}x = e^2(2e-1)$ (M1) for setting equation

$\int_2^k xe^x\mathrm{d}x = e^2(2e-1)$

Let $\theta = e^x$. (M1) for valid approach

$\frac{\mathrm{d}\theta}{\mathrm{d}x} = e^x \Rightarrow \mathrm{d}\theta = e^x\mathrm{d}x$

$\therefore \int_2^k xe^x\mathrm{d}x = \int_2^k x\mathrm{d}(e^x)$

$\left[(x)(e^x)\right]_2^k - \int_2^k e^x\mathrm{d}x = e^2(2e-1)$ (A2) for correct working

$\left[xe^x\right]_2^k - \left[e^x\right]_2^k = e^2(2e-1)$ A1

$ke^k - 2e^2 - (e^k - e^2) = 2e^3 - e^2$

$(k-1)e^k - e^2 = 2e^3 - e^2$

$(k-1)e^k = 2e^3$

$\therefore k = 3$ A1

[6]

Exercise 56

1. The volume of the solid generated

$= \int_{\frac{2\pi}{3}}^{\frac{5\pi}{6}} \pi y^2 \mathrm{d}x$ (A1) for correct approach

$= \int_{\frac{2\pi}{3}}^{\frac{5\pi}{6}} \pi \cot\left(x - \frac{\pi}{2}\right)\operatorname{cosec}\left(x - \frac{\pi}{2}\right)\mathrm{d}x$

Let $u = x - \frac{\pi}{2}$. (M1) for substitution

$\frac{\mathrm{d}u}{\mathrm{d}x} = 1 \Rightarrow \mathrm{d}u = \mathrm{d}x$

$x = \frac{5\pi}{6} \Rightarrow u = \frac{5\pi}{6} - \frac{\pi}{2} = \frac{\pi}{3}$

$x = \frac{2\pi}{3} \Rightarrow u = \frac{2\pi}{3} - \frac{\pi}{2} = \frac{\pi}{6}$

$\therefore \int_{\frac{2\pi}{3}}^{\frac{5\pi}{6}} \pi y^2 \mathrm{d}x = \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \pi \cot u \operatorname{cosec} u \mathrm{d}u$ (A2) for correct working

$\int_{\frac{2\pi}{3}}^{\frac{5\pi}{6}} \pi y^2 \mathrm{d}x = \left[-\pi \operatorname{cosec} u\right]_{\frac{\pi}{6}}^{\frac{\pi}{3}}$ A1

$\int_{\frac{2\pi}{3}}^{\frac{5\pi}{6}} \pi y^2 \mathrm{d}x = -\pi \operatorname{cosec}\frac{\pi}{3} - \left(-\pi \operatorname{cosec}\frac{\pi}{6}\right)$

$\int_{\frac{2\pi}{3}}^{\frac{5\pi}{6}} \pi y^2 \mathrm{d}x = -\frac{2\sqrt{3}\pi}{3} + 2\pi$

$\int_{\frac{2\pi}{3}}^{\frac{5\pi}{6}} \pi y^2 \mathrm{d}x = \frac{2(3 - \sqrt{3})\pi}{3}$ A1

[6]

2. $y = \ln \sqrt[\pi]{x} - \ln \sqrt[\pi]{\pi}$

$y = \ln \sqrt[\pi]{\frac{x}{\pi}}$

$y = \ln \left(\frac{x}{\pi}\right)^{\frac{1}{\pi}}$ (M1) for valid approach

$y = \frac{1}{\pi} \ln \left(\frac{x}{\pi}\right)$

$\pi y = \ln \left(\frac{x}{\pi}\right)$

$e^{\pi y} = \frac{x}{\pi}$

$\therefore x = \pi e^{\pi y}$ A1

The volume of the solid generated

$= \int_0^{2\pi} \pi x^2 \mathrm{d}y$ (A1) for correct approach

$= \int_0^{2\pi} \pi^2 e^{2\pi y} \mathrm{d}y$

Let $u = 2\pi y$. (M1) for substitution

$\frac{\mathrm{d}u}{\mathrm{d}y} = 2\pi \Rightarrow \frac{1}{2}\mathrm{d}u = \pi \mathrm{d}y$

$y = 2\pi \Rightarrow u = 2\pi(2\pi) = 4\pi^2$

$y = 0 \Rightarrow u = 2\pi(0) = 0$

$\therefore \int_0^{2\pi} \pi x^2 \mathrm{d}y = \int_0^{4\pi^2} \frac{\pi}{2} e^u \mathrm{d}u$ (A2) for correct working

$\int_0^{2\pi} \pi x^2 \mathrm{d}y = \left[\frac{\pi}{2} e^u\right]_0^{4\pi^2}$ A1

$\int_0^{2\pi} \pi x^2 \mathrm{d}y = \frac{\pi}{2} e^{4\pi^2} - \frac{\pi}{2} e^0$

$\int_0^{2\pi} \pi x^2 \mathrm{d}y = \frac{\pi}{2}(e^{4\pi^2} - 1)$ A1

[8]

3. $\dfrac{x^2}{9a^2}+\dfrac{y^2}{4a^2}=1$

$\dfrac{y^2}{4a^2}=1-\dfrac{x^2}{9a^2}$

$y^2=4a^2-\dfrac{4}{9}x^2$ A1

When $y=0$,

$\dfrac{x^2}{9a^2}+0=1$ M1

$x^2=9a^2$

$x=-3a$ or $x=3a$ A1

The volume of the rugby model

$=\int_{-3a}^{3a}\pi y^2\mathrm{d}x$ A1

$=\int_{-3a}^{3a}\pi\left(4a^2-\dfrac{4}{9}x^2\right)\mathrm{d}x$

$=\pi\left[4a^2x-\dfrac{4}{9}\left(\dfrac{1}{3}x^3\right)\right]_{-3a}^{3a}$ A1

$=\pi\left[4a^2x-\dfrac{4}{27}x^3\right]_{-3a}^{3a}$

$=\pi\left(\left(4a^2(3a)-\dfrac{4}{27}(3a)^3\right)-\left(4a^2(-3a)-\dfrac{4}{27}(-3a)^3\right)\right)$ A1

$=\pi(12a^3-4a^3+12a^3-4a^3)$

$=16\pi a^3$ AG

[6]

4. $hx + ry - 3hr = 0$

$hx = 3hr - ry$

$x = 3r - \frac{r}{h}y$ A1

The volume of the conical frustum

$= \int_0^{2h} \pi x^2 \mathrm{d}y$ A1

$= \int_0^{2h} \pi \left(3r - \frac{r}{h}y\right)^2 \mathrm{d}y$

$= \int_0^{2h} \pi \left(9r^2 - \frac{6r^2}{h}y + \frac{r^2}{h^2}y^2\right) \mathrm{d}y$ M1

$= \pi \left[9r^2 y - \frac{6r^2}{h}\left(\frac{1}{2}y^2\right) + \frac{r^2}{h^2}\left(\frac{1}{3}y^3\right)\right]_0^{2h}$ A1

$= \pi \left[9r^2 y - \frac{3r^2}{h}y^2 + \frac{r^2}{3h^2}y^3\right]_0^{2h}$ M1

$= \pi \left(\left(9r^2(2h) - \frac{3r^2}{h}(2h)^2 + \frac{r^2}{3h^2}(2h)^3\right) - 0\right)$ A1

$= \pi \left(18r^2 h - 12r^2 h + \frac{8}{3}r^2 h\right)$ A1

$= \frac{26}{3}\pi r^2 h$ AG

[7]

Exercise 57

1. (a) $\frac{2\pi}{\alpha} = 8\pi$ M1A1

$\alpha = \frac{1}{4}$ AG

[2]

(b) $f(-x) = \sec\frac{1}{4}(-x)$ M1

$f(-x) = \sec\left(-\frac{1}{4}x\right)$

$f(-x) = \frac{1}{\cos\left(-\frac{1}{4}x\right)}$

$f(-x) = \frac{1}{\cos\frac{1}{4}x}$ A1

$f(-x) = \sec\frac{1}{4}x$

$f(-x) = f(x)$

Thus, f is an even function. AG

[2]

(c) (i) $\{x : 0 \le x < 2\pi\}$ A2

(ii) $y = \sec\frac{1}{4}x$

$\Rightarrow x = \sec\frac{1}{4}y$ (M1) for swapping variables

$\frac{1}{x} = \cos\frac{1}{4}y$

$\arccos\frac{1}{x} = \frac{1}{4}y$ A1

$y = 4\arccos\frac{1}{x}$

$\therefore f^{-1}(x) = 4\arccos\frac{1}{x}$ A1

[5]

(d) $\int_{-\pi}^{a} \pi \sec^2 \frac{1}{4}x \, dx = 4\pi(\sqrt{3}+1)$ (A1) for correct approach

Let $u = \frac{1}{4}x$. (M1) for substitution

$\frac{du}{dx} = \frac{1}{4} \Rightarrow 4du = dx$

$x = a \Rightarrow u = \frac{1}{4}a$

$x = -\pi \Rightarrow u = \frac{1}{4}(-\pi) = -\frac{1}{4}\pi$

$\therefore \int_{-\frac{1}{4}\pi}^{\frac{1}{4}a} 4\pi \sec^2 u \, du = 4\pi(\sqrt{3}+1)$ (A2) for correct working

$\therefore \int_{-\frac{1}{4}\pi}^{\frac{1}{4}a} \sec^2 u \, du = \sqrt{3}+1$

$[\tan u]_{-\frac{1}{4}\pi}^{\frac{1}{4}a} = \sqrt{3}+1$ A1

$\tan\frac{1}{4}a - \tan\left(-\frac{1}{4}\pi\right) = \sqrt{3}+1$

$\tan\frac{1}{4}a - (-1) = \sqrt{3}+1$ A1

$\tan\frac{1}{4}a = \sqrt{3}$

$\frac{1}{4}a = \frac{\pi}{3}$ M1

$a = \frac{4\pi}{3}$ A1

[8]

2. (a) $f(-x)=\dfrac{1}{\sqrt{a^2-(-x)^2}}$ M1

$f(-x)=\dfrac{1}{\sqrt{a^2-x^2}}$ A1

$f(-x)=f(x)$

Thus, f is an even function. AG

[2]

(b) (i) $\{x: 0 \le x < a\}$ A2

(ii) $y=\dfrac{1}{\sqrt{a^2-x^2}}$

$\Rightarrow x=\dfrac{1}{\sqrt{a^2-y^2}}$ (M1) for swapping variables

$x^2=\dfrac{1}{a^2-y^2}$

$a^2-y^2=\dfrac{1}{x^2}$ A1

$y^2=a^2-\dfrac{1}{x^2}$

$y=\sqrt{a^2-\dfrac{1}{x^2}}$

$\therefore f^{-1}(x)=\sqrt{a^2-\dfrac{1}{x^2}}$ A1

[5]

(c) The area of R

$=\displaystyle\int_{-\frac{\sqrt{3}}{2}a}^{\frac{\sqrt{3}}{2}a}\frac{1}{\sqrt{a^2-x^2}}\,\mathrm{d}x$ (A1) for correct approach

$=\left[\arcsin\dfrac{x}{a}\right]_{-\frac{\sqrt{3}}{2}a}^{\frac{\sqrt{3}}{2}a}$ A1

$=\arcsin\dfrac{\sqrt{3}}{2}-\arcsin\left(-\dfrac{\sqrt{3}}{2}\right)$ (M1) for substitution

$=\dfrac{\pi}{3}-\left(-\dfrac{\pi}{3}\right)$ A1

$=\dfrac{2\pi}{3}$ A1

[5]

(d) The volume of the solid generated

$$= \int_{-\frac{\sqrt{2}}{2}a}^{\frac{\sqrt{2}}{2}a} \pi y^2 \mathrm{d}x$$

$$= \int_{-\frac{\sqrt{2}}{2}a}^{\frac{\sqrt{2}}{2}a} \pi \left(\frac{1}{\sqrt{a^2 - x^2}} \right)^2 \mathrm{d}x \qquad \text{A1}$$

$$= \pi \int_{-\frac{\sqrt{2}}{2}a}^{\frac{\sqrt{2}}{2}a} \frac{1}{a^2 - x^2} \mathrm{d}x$$

Let $x = a\sin\theta$. A1

$$\frac{\mathrm{d}x}{\mathrm{d}\theta} = a\cos\theta \Rightarrow \mathrm{d}x = a\cos\theta \mathrm{d}\theta$$

$$x = \frac{\sqrt{2}}{2}a \Rightarrow \sin\theta = \frac{\sqrt{2}}{2} \Rightarrow \theta = \frac{\pi}{4} \qquad \text{M1}$$

$$x = -\frac{\sqrt{2}}{2}a \Rightarrow \sin\theta = -\frac{\sqrt{2}}{2} \Rightarrow \theta = -\frac{\pi}{4}$$

$$\therefore \int_{-\frac{\sqrt{2}}{2}a}^{\frac{\sqrt{2}}{2}a} \pi y^2 \mathrm{d}x = \pi \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{1}{a^2 - a^2\sin^2\theta} \cdot a\cos\theta \mathrm{d}\theta \qquad \text{A1}$$

$$\int_{-\frac{\sqrt{2}}{2}a}^{\frac{\sqrt{2}}{2}a} \pi y^2 \mathrm{d}x = \pi \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{a\cos\theta}{a^2(1 - \sin^2\theta)} \mathrm{d}\theta$$

$$\int_{-\frac{\sqrt{2}}{2}a}^{\frac{\sqrt{2}}{2}a} \pi y^2 \mathrm{d}x = \pi \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{a\cos\theta}{a^2\cos^2\theta} \mathrm{d}\theta \qquad \text{A1}$$

$$\int_{-\frac{\sqrt{2}}{2}a}^{\frac{\sqrt{2}}{2}a} \pi y^2 \mathrm{d}x = \pi \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{1}{a\cos\theta} \mathrm{d}\theta \qquad \text{M1}$$

$$\int_{-\frac{\sqrt{2}}{2}a}^{\frac{\sqrt{2}}{2}a} \pi y^2 \mathrm{d}x = \frac{\pi}{a} \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \sec\theta \mathrm{d}\theta$$

$$\therefore \int_{-\frac{\sqrt{2}}{2}a}^{\frac{\sqrt{2}}{2}a} \pi y^2 \mathrm{d}x = \frac{\pi}{a} \Big[\ln|\sec\theta + \tan\theta| \Big]_{-\frac{1}{4}\pi}^{\frac{1}{4}\pi} \qquad \text{A1}$$

$$\int_{-\frac{\sqrt{2}}{2}a}^{\frac{\sqrt{2}}{2}a} \pi y^2 \mathrm{d}x = \frac{\pi}{a} \left(\begin{array}{l} \ln\left|\sec\frac{1}{4}\pi + \tan\frac{1}{4}\pi\right| \\ -\ln\left|\sec\left(-\frac{1}{4}\pi\right) + \tan\left(-\frac{1}{4}\pi\right)\right| \end{array} \right)$$

$$\int_{-\frac{\sqrt{2}}{2}a}^{\frac{\sqrt{2}}{2}a} \pi y^2 \mathrm{d}x = \frac{\pi}{a} (\ln(\sqrt{2} + 1) - \ln(\sqrt{2} - 1)) \qquad \text{A2}$$

$$\int_{-\frac{\sqrt{2}}{2}a}^{\frac{\sqrt{2}}{2}a} \pi y^2 \mathrm{d}x = \frac{\pi}{a} \ln\frac{\sqrt{2} + 1}{\sqrt{2} - 1} \qquad \text{A1}$$

$$\int_{-\frac{\sqrt{2}}{2}a}^{\frac{\sqrt{2}}{2}a} \pi y^2 \mathrm{d}x = \frac{\pi}{a} \ln \frac{(\sqrt{2}+1)(\sqrt{2}+1)}{(\sqrt{2}-1)(\sqrt{2}+1)}$$ M1

$$\int_{-\frac{\sqrt{2}}{2}a}^{\frac{\sqrt{2}}{2}a} \pi y^2 \mathrm{d}x = \frac{\pi}{a} \ln \frac{2+2\sqrt{2}+1}{2-1}$$ A1

$$\int_{-\frac{\sqrt{2}}{2}a}^{\frac{\sqrt{2}}{2}a} \pi y^2 \mathrm{d}x = \frac{\pi}{a} \ln(3+2\sqrt{2})$$ AG

[12]

3. (a) If $f(a) = f(b)$,

$2 \cdot 3^a = 2 \cdot 3^b$ M1

$3^a = 3^b$

$a = b$ R1

Thus, f is a one-to-one function. AG

[2]

(b) $y = 2 \cdot 3^x$

$\Rightarrow x = 2 \cdot 3^y$ (M1) for swapping variables

$\frac{x}{2} = 3^y$

$y = \log_3 \frac{x}{2}$

$\therefore f^{-1}(x) = \log_3 \frac{x}{2}$ A1

[2]

(c) $f'(x) = 2 \cdot 3^x \ln 3$ (A1) for correct approach

$f'(a) = \frac{2 \cdot 3^a - 0}{a - 0}$ (M1) for setting equation

$\therefore 2 \cdot 3^a \ln 3 = \frac{2 \cdot 3^a}{a}$ (A1) for substitution

$\ln 3 = \frac{1}{a}$

$a = \frac{1}{\ln 3}$ A1

The equation of L:

$y - 0 = \frac{2 \cdot 3^{\frac{1}{\ln 3}}}{\frac{1}{\ln 3}}(x - 0)$ A1

$y = \left(2 \ln 3 \cdot 3^{\frac{1}{\ln 3}} \right) x$ A1

[6]

(d) The volume of the solid generated

$$= \int_0^{\frac{1}{\ln 3}} \pi(2\cdot 3^x)^2 \mathrm{d}x - \int_0^{\frac{1}{\ln 3}} \pi\left(\left(2\ln 3\cdot 3^{\frac{1}{\ln 3}}\right)x\right)^2 \mathrm{d}x$$ M1A1

$$= \pi\int_0^{\frac{1}{\ln 3}}\left(4\cdot 9^x - 4(\ln 3)^2\cdot 3^{\frac{2}{\ln 3}}x^2\right)\mathrm{d}x$$ A1

$$= \pi\left[\frac{4\cdot 9^x}{\ln 9} - \frac{4}{3}(\ln 3)^2\cdot 3^{\frac{2}{\ln 3}}x^3\right]_0^{\frac{1}{\ln 3}}$$ A1

$$= \pi\left(\frac{4\cdot 9^{\frac{1}{\ln 3}}}{\ln 9} - \frac{4}{3}(\ln 3)^2\cdot 3^{\frac{2}{\ln 3}}\left(\frac{1}{\ln 3}\right)^3 - \left(\frac{4\cdot 9^0}{\ln 9} - 0\right)\right)$$

$$= \pi\left(\frac{4\cdot 9^{\frac{1}{\ln 3}}}{\ln 9} - \frac{4\cdot 3^{\frac{2}{\ln 3}}}{3\ln 3} - \frac{4}{\ln 9}\right)$$ M1

$$= \pi\left(\frac{4\cdot 9^{\frac{1}{\ln 3}}}{\ln 9} - \frac{4\cdot 9^{\frac{1}{\ln 3}}}{3\ln 3} - \frac{4}{\ln 9}\right)$$

$$= \pi\left(\frac{4\cdot 9^{\frac{1}{\ln 3}}}{2\ln 3} - \frac{4\cdot 9^{\frac{1}{\ln 3}}}{3\ln 3} - \frac{4}{2\ln 3}\right)$$ A1

$$= \pi\left(\frac{12\cdot 9^{\frac{1}{\ln 3}}}{6\ln 3} - \frac{8\cdot 9^{\frac{1}{\ln 3}}}{6\ln 3} - \frac{12}{6\ln 3}\right)$$

$$= \pi\left(\frac{12\cdot 9^{\frac{1}{\ln 3}} - 8\cdot 9^{\frac{1}{\ln 3}} - 12}{6\ln 3}\right)$$ M1

$$= \pi\left(\frac{4\cdot 9^{\frac{1}{\ln 3}} - 12}{6\ln 3}\right)$$ M1

$$= \pi\left(\frac{4\left(9^{\frac{1}{\ln 3}} - 3\right)}{6\ln 3}\right)$$

$$= \frac{2\pi\left(9^{\frac{1}{\ln 3}} - 3\right)}{3\ln 3}$$ AG

[8]

4. (a) If $f(a) = f(b)$,

$k\log_2 a = k\log_2 b$ M1

$\log_2 a = \log_2 b$

$a = b$ A1

Thus, f is a one-to-one function. AG

[2]

(b) $y = k\log_2 x$

$\Rightarrow x = k\log_2 y$ (M1) for swapping variables

$\frac{x}{k} = \log_2 y$

$y = 2^{\frac{x}{k}}$

$\therefore f^{-1}(x) = 2^{\frac{x}{k}}$ A1

[2]

(c) $\frac{1}{32} = 2^{-\frac{5}{k}}$ (M1) for setting equation

$2^{-5} = 2^{-\frac{5}{k}}$

$-5 = -\frac{5}{k}$

$k = 1$ A1

[2]

(d) $0 = \log_2 x$

$x = 1$ (A1) for correct value

The area of R

$= \int_1^{e^2} \log_2 x \mathrm{d}x$ (A1) for correct approach

$= \int_1^{e^2} \frac{\ln x}{\ln 2} \mathrm{d}x$ A1

$= \frac{1}{\ln 2} \int_1^{e^2} \ln x \mathrm{d}x$

$= \frac{1}{\ln 2}\left([x \ln x]_1^{e^2} - \int_1^{e^2} x \mathrm{d}(\ln x) \right)$ (A2) for correct working

$= \frac{1}{\ln 2}\left([x \ln x]_1^{e^2} - \int_1^{e^2} x \cdot \frac{1}{x} \mathrm{d}x \right)$

$= \frac{1}{\ln 2}\left([x \ln x]_1^{e^2} - \int_1^{e^2} \mathrm{d}x \right)$ (M1) for valid approach

$= \frac{1}{\ln 2}\left([x \ln x]_1^{e^2} - [x]_1^{e^2} \right)$ A1

$= \frac{1}{\ln 2}[x \ln x - x]_1^{e^2}$

$= \frac{1}{\ln 2}\left[(e^2 \ln e^2 - e^2) - (1 \ln 1 - 1) \right]_1^{e^2}$ A1

$= \frac{1}{\ln 2}(2e^2 - e^2 + 1)$

$= \frac{e^2 + 1}{\ln 2}$ A1

[9]

(e) $\int_0^b 2^x \mathrm{d}x = \frac{e^2 + 1}{\ln 2}$ M1A1

$\left[\frac{2^x}{\ln 2} \right]_0^b = \frac{e^2 + 1}{\ln 2}$ A1

$\frac{2^b}{\ln 2} - \frac{2^0}{\ln 2} = \frac{e^2 + 1}{\ln 2}$

$2^b - 1 = e^2 + 1$ A1

$2^b = e^2 + 2$

$b = \log_2(e^2 + 2)$ AG

[4]

Exercise 58

1. (a) $\log_8 y^3 + \log_2 x = \log_8 16\sqrt{2}$

$\log_8 y^3 + \dfrac{\log_8 x}{\log_8 2} = \log_8 16\sqrt{2}$ (A1) for correct formula

$\log_8 y^3 + \dfrac{\log_8 x}{\frac{1}{3}} = \log_8 16\sqrt{2}$ (A1) for correct value

$\log_8 y^3 + 3\log_8 x = \log_8 16\sqrt{2}$ (M1) for valid approach

$\log_8 y^3 + \log_8 x^3 = \log_8 16\sqrt{2}$ (A1) for correct approach

$\log_8 y^3 x^3 = \log_8 16\sqrt{2}$ (A1) for correct formula

$\therefore y^3 x^3 = 16\sqrt{2}$ M1

$yx = 2\sqrt{2}$

$x = 2\sqrt{2}y^{-1}$ A1

[7]

(b) $\int_1^c 2\sqrt{2}y^{-1}\mathrm{d}y = 2\sqrt{2}$ M1A1

$\left[2\sqrt{2}\ln y\right]_1^c = 2\sqrt{2}$ A1

$2\sqrt{2}\ln c - 2\sqrt{2}\ln 1 = 2\sqrt{2}$

$2\sqrt{2}\ln c = 2\sqrt{2}$ A1

$\ln c = 1$ M1

$c = e$ AG

[5]

(c) $\int_\alpha^{2\alpha}(f(y)-g(y))\mathrm{d}y = \int_\alpha^{2\alpha}(f(y)-(-f(y)))\mathrm{d}y$ A1

$\int_\alpha^{2\alpha}(f(y)-g(y))\mathrm{d}y = \int_\alpha^{2\alpha}2f(y)\mathrm{d}y$

$\int_\alpha^{2\alpha}(f(y)-g(y))\mathrm{d}y = \int_\alpha^{2\alpha}2(2\sqrt{2}y^{-1})\mathrm{d}y$ M1

$\int_\alpha^{2\alpha}(f(y)-g(y))\mathrm{d}y = \left[4\sqrt{2}\ln y\right]_\alpha^{2\alpha}$ A1

$\int_\alpha^{2\alpha}(f(y)-g(y))\mathrm{d}y = 4\sqrt{2}\ln 2\alpha - 4\sqrt{2}\ln\alpha$

$\int_\alpha^{2\alpha}(f(y)-g(y))\mathrm{d}y = 4\sqrt{2}\ln\dfrac{2\alpha}{\alpha}$ A1

$\int_\alpha^{2\alpha}(f(y)-g(y))\mathrm{d}y = 4\sqrt{2}\ln 2$ AG

[4]

(d) $\int_{125}^{8000} (f(y)-g(y))\mathrm{d}y$

$= \int_{125}^{250} (f(y)-g(y))\mathrm{d}y + \int_{250}^{500} (f(y)-g(y))\mathrm{d}y$

$+\int_{500}^{1000} (f(y)-g(y))\mathrm{d}y + \int_{1000}^{2000} (f(y)-g(y))\mathrm{d}y$ (A1) for correct approach

$+\int_{2000}^{4000} (f(y)-g(y))\mathrm{d}y + \int_{4000}^{8000} (f(y)-g(y))\mathrm{d}y$

$= \int_{125}^{2(125)} (f(y)-g(y))\mathrm{d}y + \int_{250}^{2(250)} (f(y)-g(y))\mathrm{d}y$

$+\int_{500}^{2(500)} (f(y)-g(y))\mathrm{d}y + \int_{1000}^{2(1000)} (f(y)-g(y))\mathrm{d}y$ (M1) for valid approach

$+\int_{2000}^{2(2000)} (f(y)-g(y))\mathrm{d}y + \int_{4000}^{2(4000)} (f(y)-g(y))\mathrm{d}y$

$= 6(4\sqrt{2}\ln 2)$ A1

$= 24\sqrt{2}\ln 2$ A1

[4]

2. (a) $f(x) = g(x)$

$4e^{2x} = 4e^{x+2}$ — (M1) for setting equation

$e^{2x} = e^{x+2}$

$e^{x} = e^{2}$ — (A1) for correct approach

$x = 2$ — (A1) for correct value

$f(2) = 4e^{2(2)}$ — (M1) for substitution

$f(2) = 4e^{4}$

Thus, the required coordinates are $(2, 4e^{4})$. — A1

[5]

(b) The area of R

$= \int_0^2 (g(x) - f(x))\mathrm{d}x$ — A1

$= \int_0^2 (4e^{x+2} - 4e^{2x})\mathrm{d}x$

$= \left[4e^{x+2} - \frac{4}{2}e^{2x} \right]_0^2$ — A1

$= \left[4e^{x+2} - 2e^{2x} \right]_0^2$

$= (4e^{2+2} - 2e^{2(2)}) - (4e^{0+2} - 2e^{2(0)})$ — M1

$= (4e^{4} - 2e^{4}) - (4e^{2} - 2)$

$= 2e^{4} - 4e^{2} + 2$ — M1

$= 2((e^{2})^{2} - 2e^{2} + 1)$ — A1

$= 2(e^{2} - 1)^{2}$ — A1

$= 2((e+1)(e-1))^{2}$ — M1

$= 2(e+1)^{2}(e-1)^{2}$ — AG

[7]

(c) $f'(x) = 4(e^{2x})(2)$

$f'(x) = 8e^{2x}$

$f''(x) = 8(e^{2x})(2)$

$f''(x) = 16e^{2x}$ A1

$g'(x) = 4e^{x+2}$

$g''(x) = 4e^{x+2}$ A1

$f''(x) = g''(x)$

$\therefore 16e^{2x} = 4e^{x+2}$ M1

$4e^{2x} = e^{x+2}$

$4e^{x} = e^{2}$

$e^{x} = \frac{e^2}{4}$ A1

$x = \ln\frac{e^2}{4}$ A1

$x = \ln e^2 - \ln 4$

$\therefore a = 2 - \ln 4$ AG

[5]

(d) $\text{PQ} = |f(2-\ln 4) - g(2-\ln 4)|$ (M1) for valid approach

$\text{PQ} = \left|4e^{2(2-\ln 4)} - 4e^{2-\ln 4+2}\right|$

$\text{PQ} = \left|4e^{4-2\ln 4} - 4e^{4-\ln 4}\right|$

$\text{PQ} = \left|4e^{4-2\ln 4}(1 - e^{\ln 4})\right|$ (A1) for correct approach

$\text{PQ} = \left|\frac{4e^4}{e^{\ln 16}}(1 - e^{\ln 4})\right|$

$\text{PQ} = \left|\frac{4e^4}{16}(1-4)\right|$ (A1) for correct approach

$\text{PQ} = \left|\frac{e^4}{4}(-3)\right|$

$\text{PQ} = \frac{3}{4}e^4$ A1

[4]

3. (a) (i) $f(x)=g(x)$

$\therefore \sin \pi x=\sin 2\pi x$ (M1) for setting equation

$\sin \pi x=2\sin \pi x\cos \pi x$ (A1) for substitution

$\sin \pi x-2\sin \pi x\cos \pi x=0$

$\sin \pi x(1-2\cos \pi x)=0$ (A1) for factorization

$\sin \pi x=0$ or $\cos \pi x=\frac{1}{2}$

$\pi x=0$, $\pi x=\pi$ or $\pi x=\frac{\pi}{3}$ A1

$x=0$ (*Rejected*), $x=1$ (*Rejected*) or $x=\frac{1}{3}$

$\therefore r=\frac{1}{3}$ A1

(ii) The area of the region

$=\int_{\frac{1}{3}}^{1}(f(x)-g(x))\mathrm{d}x$ A1

$=\int_{\frac{1}{3}}^{1}(\sin \pi x-\sin 2\pi x)\mathrm{d}x$

$=\left[-\frac{1}{\pi}\cos \pi x+\frac{1}{2\pi}\cos 2\pi x\right]_{\frac{1}{3}}^{1}$ A1

$=\left(-\frac{1}{\pi}\cos \pi(1)+\frac{1}{2\pi}\cos 2\pi(1)\right)$

$-\left(-\frac{1}{\pi}\cos \pi\left(\frac{1}{3}\right)+\frac{1}{2\pi}\cos 2\pi\left(\frac{1}{3}\right)\right)$ M1

$=\left(-\frac{1}{\pi}(-1)+\frac{1}{2\pi}(1)\right)$

$-\left(-\frac{1}{\pi}\left(\frac{1}{2}\right)+\frac{1}{2\pi}\left(-\frac{1}{2}\right)\right)$ A1

$=\frac{1}{\pi}+\frac{1}{2\pi}+\frac{1}{2\pi}+\frac{1}{4\pi}$ M1

$=\frac{4}{4\pi}+\frac{2}{4\pi}+\frac{2}{4\pi}+\frac{1}{4\pi}$

$=\frac{9}{4\pi}$ AG

[10]

(b)	The coordinates of Q are $\left(\frac{1}{4}, 1\right)$.	(A1) for correct values
	$\therefore a\sin\pi\left(\frac{1}{4}\right)=1$	(M1) for substitution
	$\frac{\sqrt{2}}{2}a=1$	A1
	$a=\sqrt{2}$	A1

[4]

(c)	$f(x)>g(x)$	
	$\therefore a\sin\pi x>\sin 2\pi x$	(M1) for setting inequality
	$a\sin\pi x>2\sin\pi x\cos\pi x$	
	$a>2\cos\pi x$	(A1) for correct approach
	$0<x<1$	
	$0<\pi x<\pi$	
	$-1<\cos\pi x<1$	(A1) for correct values
	$\therefore 2\cos\pi x<2$	
	Thus, the least possible value of a is 2.	A1

[4]

4. (a) (i) $f(x)=g(x)$

$\therefore \cos 2\pi y=\cos \pi y$ M1

$2\cos^2 \pi y-1=\cos \pi y$ A1

$2\cos^2 \pi y-\cos \pi y-1=0$ A1

$(2\cos \pi y+1)(\cos \pi y-1)=0$ A1

$\cos \pi y=-\dfrac{1}{2}$ or $\cos \pi y=1$

$\pi y=\dfrac{2\pi}{3}$ or $\pi y=0$ A1

$y=\dfrac{2}{3}$ or $y=0$ (*Rejected*)

$\therefore r=\dfrac{2}{3}$ AG

(ii) The area of the region

$=\int_0^{\frac{2}{3}}(g(y)-f(y))\mathrm{d}y$ A1

$=\int_0^{\frac{2}{3}}(\cos \pi y-\cos 2\pi y)\mathrm{d}y$

$=\left[\dfrac{1}{\pi}\sin \pi y-\dfrac{1}{2\pi}\sin 2\pi y\right]_0^{\frac{2}{3}}$ A1

$=\left(\dfrac{1}{\pi}\sin \pi\left(\dfrac{2}{3}\right)-\dfrac{1}{2\pi}\sin 2\pi\left(\dfrac{2}{3}\right)\right)$

$-\left(\dfrac{1}{\pi}\sin \pi(0)-\dfrac{1}{2\pi}\sin 2\pi(0)\right)$ M1

$=\left(\dfrac{1}{\pi}\left(\dfrac{\sqrt{3}}{2}\right)-\dfrac{1}{2\pi}\left(-\dfrac{\sqrt{3}}{2}\right)\right)-0$ A1

$=\dfrac{\sqrt{3}}{2\pi}+\dfrac{\sqrt{3}}{4\pi}$ M1

$=\dfrac{2\sqrt{3}}{4\pi}+\dfrac{\sqrt{3}}{4\pi}$

$=\dfrac{3\sqrt{3}}{4\pi}$ AG

[10]

(b) $a\cos 2\pi\left(\frac{1}{6}\right)=\frac{\sqrt{3}}{2}$ (M1) for substitution

$\frac{1}{2}a=\frac{\sqrt{3}}{2}$ A1

$a=\sqrt{3}$ A1

[3]

(c) $f(x)=g(x)$

$\therefore a\cos 2\pi y=\cos \pi y$ M1

$a(2\cos^2 \pi y-1)=\cos \pi y$ A1

$2a\cos^2 \pi y-a=\cos \pi y$

$2a\cos^2 \pi y-\cos \pi y-a=0$ A1

$\cos \pi y=\frac{-(-1)\pm\sqrt{(-1)^2-4(2a)(-a)}}{2(2a)}$ M1A1

$\cos \pi y=\frac{1\pm\sqrt{1+8a^2}}{4a}$

$\cos \pi y=\frac{1+\sqrt{1+8a^2}}{4a}$ or

$\cos \pi y=\frac{1-\sqrt{1+8a^2}}{4a}$ (*Rejected*) A1

$\pi y=\arccos\left(\frac{1+\sqrt{1+8a^2}}{4a}\right)$ M1

$\therefore r=\frac{1}{\pi}\arccos\left(\frac{1+\sqrt{1+8a^2}}{4a}\right)$ AG

[7]

Exercise 59

1. (a) $g(x)=\ln(x-7)^3$

$g(x)=3\ln(x-7)$ (M1) for valid approach

$g(x)=3f(x-7)$ (M1) for valid approach

$\therefore p=7,\ q=3$ A2

[4]

(b) The volume generated

$=\int_{3\pi}^{10}\pi(g(x))^2\,\mathrm{d}x$

$=\int_{3\pi}^{10}\pi(\ln(x-7)^3)^2\,\mathrm{d}x$ (A1) for correct approach

$=16.19380939$

$=16.2$ A1

[2]

2. (a) $g(x)=2^x$

$g(x)=8^{\frac{1}{3}x}-5+5$ (M1) for valid approach

$g(x)=f\left(\frac{1}{3}x\right)+5$ (M1) for valid approach

$\therefore p=\frac{1}{3},\ q=5$ A2

[4]

(b) The volume generated

$=\int_{0}^{\pi}\pi(g(x))^2\,\mathrm{d}x$

$=\int_{0}^{\pi}\pi(2^x)^2\,\mathrm{d}x$ (A1) for correct approach

$=174.2244533$

$=174$ A1

[2]

3. $y = \sec \pi(0)$

$y = 1$ (A1) for correct value

$y = \sec \pi x$

$y = \dfrac{1}{\cos \pi x}$ (M1) for valid approach

$\cos \pi x = \dfrac{1}{y}$

$\pi x = \arccos \dfrac{1}{y}$ (M1) for valid approach

$x = \dfrac{1}{\pi} \arccos \dfrac{1}{y}$ A1

The volume generated

$= \int_1^e \pi \left(\dfrac{1}{\pi} \arccos \dfrac{1}{y} \right)^2 \mathrm{d}y$ (A1) for correct approach

$= 0.5007292734$

$= 0.501$ A1

[6]

4. When $x = 0$,

$y = e^{4(0)}$

$y = 1$ (A1) for correct value

$y = e^{4x}$

$\ln y = 4x$ (M1) for valid approach

$x = \dfrac{1}{4} \ln y$ A1

The volume generated

$= \int_1^e \pi \left(\dfrac{1}{4} \ln y \right)^2 \mathrm{d}y$ (A1) for correct approach

$= 0.1410343072$

$= 0.141$ A1

[5]

Exercise 60

1. $0 = e^{2x} - 1$

$1 = e^{2x}$

$2x = 0$

$x = 0$ (A1) for correct value

$x + y - e^{6} - 2 = 0$

$y = -x + e^{6} + 2$ (A1) for correct approach

$e^{2x} - 1 = -x + e^{6} + 2$ (M1) for setting equation

$e^{2x} - 1 = e^{6} - (x - 2)$

$2x = 6$

$x = 3$ (A1) for correct value

$x + 0 - e^{6} - 2 = 0$

$x = e^{6} + 2$ (A1) for correct value

The area of R

$= \int_{0}^{3} (e^{2x} - 1)\mathrm{d}x + \int_{3}^{e^{6}+2} (-x + e^{6} + 2)\mathrm{d}x$ A1

$= 81172.68131$

$= 81200$ A1

[7]

2. $0 = 2\ln x$

$\ln x = 0$

$x = 1$ (A1) for correct value

$4x + e^2 y - 8e^2 = 0$

$e^2 y = 8e^2 - 4x$

$y = 8 - \frac{4}{e^2}x$ (A1) for correct approach

$2\ln x = 8 - \frac{4}{e^2}x$ (M1) for setting equation

$2\ln x + \frac{4}{e^2}x - 8 = 0$

By considering the graph of $y = 2\ln x + \frac{4}{e^2}x - 8$,

$x = 7.3890561$. (A1) for correct value

$4x + e^2(0) - 8e^2 = 0$

$4x = 8e^2$

$x = 2e^2$ (A1) for correct value

The volume of the solid

$= \int_1^{7.3890561} \pi(2\ln x)^2 \, dx + \int_{7.3890561}^{2e^2} \pi\left(8 - \frac{4}{e^2}x\right)^2 dx$ A1

$= 284.3793169$

$= 284$ A1

[7]

3. $y = 5^{-x}$

$\log_5 y = -x$

$x = -\log_5 y$ (A1) for correct approach

$y = 5^{-0}$

$y = 1$ (A1) for correct value

$y = 25x + 75$

$y - 75 = 25x$

$x = \frac{1}{25}y - 3$ (A1) for correct approach

$-\log_5 y = \frac{1}{25}y - 3$ (M1) for setting equation

$\frac{1}{25}y - 3 + \log_5 y = 0$

By considering the graph of $x = \frac{1}{25}y - 3 + \log_5 y$,

$y = 25$. (A1) for correct value

$y = 25(0) + 75$

$y = 75$ (A1) for correct value

The area of R

$= -\int_1^{25}(-\log_5 y)\mathrm{d}y - \int_{25}^{75}\left(\frac{1}{25}y - 3\right)\mathrm{d}y$ A1

$= 85.08796157$

$= 85.1$ A1

[8]

4. $y = \frac{1}{2}x + 1$

$2y = x + 2$

$x = 2y - 2$ (A1) for correct approach

$y = \frac{1}{2}(0) + 1$

$y = 1$ (A1) for correct value

$y = (x-4)^2 + 3$

$y - 3 = (x-4)^2$

$x - 4 = \sqrt{y-3}$ or $x - 4 = -\sqrt{y-3}$

$x = 4 + \sqrt{y-3}$ (*Rejected*) or $x = 4 - \sqrt{y-3}$ (A1) for correct approach

$2y - 2 = 4 - \sqrt{y-3}$ (M1) for setting equation

$2y - 6 = -\sqrt{y-3}$

$(2y-6)^2 = y - 3$

$4y^2 - 24y + 36 = y - 3$

$4y^2 - 25y + 39 = 0$ (A1) for correct approach

$(y-3)(4y-13) = 0$

$y = 3$ or $y = \frac{13}{4}$ (*Rejected*) (A1) for correct value

$y = (0-4)^2 + 3$

$y = 19$ (A1) for correct value

The volume of the solid

$= \int_1^3 \pi(2y-2)^2 \mathrm{d}y + \int_3^{19} \pi(4-\sqrt{y-3})^2 \mathrm{d}y$ A1

$= 167.551608$

$= 168$ A1

[9]

Exercise 61

1. (a) The initial velocity

$= v(0)$ (M1) for valid approach

$= (12 - 1.5(0))^3$

$= 1728 \text{ ms}^{-1}$ A1

[2]

(b) $v(t) = 3.375$ (M1) for setting equation

$(12 - 1.5t)^3 = 3.375$

$12 - 1.5t = 1.5$ (A1) for correct approach

$-1.5t = -10.5$

$t = 7$ A1

[3]

(c) The total distance travelled

$= \int_7^{11} |v(t)| \, dt$ (M1) for valid approach

$= \int_7^{11} |(12 - 1.5t)^3| \, dt$ (A1) for substitution

$= 69.1875 \text{ m}$ A1

[3]

(d) $a(t) = v'(t)$ M1

$a(t) = 3(12 - 1.5t)^2(-1.5)$ A2

$a(t) = -4.5(12 - 1.5t)^2$ AG

[3]

(e) $v(t) < a(t)$

$(12 - 1.5t)^3 < -4.5(12 - 1.5t)^2$

$(12 - 1.5t)^3 + 4.5(12 - 1.5t)^2 < 0$ (M1) for setting inequality

By considering the graph of

$y = (12 - 1.5t)^3 + 4.5(12 - 1.5t)^2$, $11 < t \le 16$. A1

[2]

(f) $a(t) = -4.5(12 - 1.5t)^2$

$$\frac{dv}{dt} = -4.5(12 - 1.5t)^2$$

$$\frac{dv}{ds} \cdot \frac{ds}{dt} = -4.5(12 - 1.5t)^2$$ A1

$$\frac{dv}{ds} \cdot (12 - 1.5t)^3 = -4.5(12 - 1.5t)^2$$ M1

$$\frac{dv}{ds} = -\frac{4.5(12 - 1.5t)^2}{(12 - 1.5t)^3}$$ A1

$$\frac{dv}{ds} = -\frac{4.5}{12 - 1.5t}\ \mathrm{s}^{-1}$$ AG

[3]

2. (a) $v(t)=0.01$ (M1) for setting equation

$-1+\ln|2+\cos t|=0.01$

$\ln|2+\cos t|=1.01$

$2+\cos t=e^{1.01}$ (A1) for correct approach

$\cos t=e^{1.01}-2$

$t=0.7293600469$

$t=0.729$ A1

[3]

(b) $|v(t)|<0.05$

$|-1+\ln|2+\cos t||<0.05$

$|-1+\ln|2+\cos t||-0.05<0$ (M1) for setting inequality

By considering the graph of

$y=|-1+\ln|2+\cos t||-0.05$,

$0.540112<t<0.945041$. (M1) for valid approach

$\therefore 0.540<t<0.945$ A1

[3]

(c) The total distance travelled

$=\int_0^1|v(t)|\,\mathrm{d}t$ (M1) for valid approach

$=\int_0^1|-1+\ln|2+\cos t||\,\mathrm{d}t$ (A1) for substitution

$=0.0580494843\text{ m}$

$=0.0580\text{ m}$ A1

[3]

(d) $a(t)=v'(t)$ (M1) for valid approach

$a(t)=0+\left(\dfrac{1}{2+\cos t}\right)(-\sin t)$ (A1) for chain rule

$a(t)=-\dfrac{\sin t}{2+\cos t}$ A1

$v(t)=0$

$-1+\ln|2+\cos t|=0$

By considering the graph of $y=-1+\ln|2+\cos t|$,

$t=0.7694667$. (A1) for correct value

The required acceleration

$=a(0.7694667)$

$=-\dfrac{\sin 0.7694667}{2+\cos 0.7694667}$ (M1) for substitution

$=-0.2559529603$

$=-0.256\text{ ms}^{-2}$ A1

[6]

(e) $v(t)\cdot a(t)<\dfrac{1}{2}$

$(-1+\ln|2+\cos t|)\left(-\dfrac{\sin t}{2+\cos t}\right)<\dfrac{1}{2}$

$(-1+\ln|2+\cos t|)\left(-\dfrac{\sin t}{2+\cos t}\right)-\dfrac{1}{2}<0$

$(-1+\ln|2+\cos t|)\left(\dfrac{\sin t}{2+\cos t}\right)+\dfrac{1}{2}>0$ (M1) for setting inequality

By considering the graph of

$y=(-1+\ln|2+\cos t|)\left(\dfrac{\sin t}{2+\cos t}\right)+\dfrac{1}{2}$,

$t>0.9118124$. (M1) for valid approach

$\therefore 0.912<t\le 1$ A1

[3]

3. (a) When $0 \le t \le 1$,

$s(t) = \int -\frac{1}{10} t \, dt$ (M1) for valid approach

$s(t) = -\frac{1}{20} t^2 + C$ (A1) for correct value

$s(0) = 1$

$\therefore -\frac{1}{20}(0)^2 + C = 1$

$C = 1$

$s(1) = -\frac{1}{20}(1)^2 + 1$

$s(1) = \frac{19}{20}$ (A1) for correct value

When $1 < t \le 3$,

$s(t) = \int \left(\frac{1}{10} t^2 - \frac{1}{5} t \right) dt$ (M1) for valid approach

$s(t) = \frac{1}{30} t^3 - \frac{1}{10} t^2 + D$ (A1) for correct value

$s(1) = \frac{19}{20}$

$\therefore \frac{1}{30}(1)^3 - \frac{1}{10}(1)^2 + D = \frac{19}{20}$

$D = \frac{61}{60}$ (A1) for correct value

$s(3) = \frac{1}{30}(3)^3 - \frac{1}{10}(3)^2 + \frac{61}{60}$

$s(3) = \frac{61}{60}$

$$\therefore s(t) = \begin{cases} -\frac{1}{20} t^2 + 1 & 0 \le t \le 1 \\ \frac{1}{30} t^3 - \frac{1}{10} t^2 + \frac{61}{60} & 1 < t \le 3 \\ \frac{61}{60} & t > 3 \end{cases}$$ A1

[7]

(b) $s(t)=1$

$\therefore \frac{1}{30}t^3-\frac{1}{10}t^2+\frac{61}{60}=1$

$\frac{1}{30}t^3-\frac{1}{10}t^2+\frac{1}{60}=0$ (M1) for setting equation

By considering the graph of $y=\frac{1}{30}t^3-\frac{1}{10}t^2+\frac{1}{60}$,

$t=2.9422419$. (M1) for valid approach

$\therefore t=0$ or $t=2.94$ A1

[3]

(c) $a(t)=\begin{cases} -\frac{1}{10}(1) & 0\le t\le 1 \\ \frac{1}{10}(2t)-\frac{1}{5}(1) & 1<t\le 3 \\ 0 & \text{otherwise} \end{cases}$ (M1) for valid approach

$a(t)=\begin{cases} -\frac{1}{10} & 0\le t\le 1 \\ \frac{1}{5}t-\frac{1}{5} & 1<t\le 3 \\ 0 & \text{otherwise} \end{cases}$ (A1) for correct values

$a(t)<\frac{3}{10}$

$\frac{1}{5}t-\frac{1}{5}<\frac{3}{10}$ (M1) for setting inequality

$\frac{1}{5}t<\frac{1}{2}$

$t<\frac{5}{2}$

$\therefore 0\le t<\frac{5}{2}$ A1

[4]

(d) $v = \frac{1}{10}t^2 - \frac{1}{5}t$

$10v = t^2 - 2t$

$10v + 1 = t^2 - 2t + 1$ A1

$10v + 1 = (t-1)^2$

$t - 1 = \sqrt{10v+1}$

$t = 1 + \sqrt{10v+1}$ A1

$\frac{dt}{dv} = 0 + \left(\frac{1}{2\sqrt{10v+1}}\right)(10)$ M1A1

$\therefore \frac{dt}{dv} = \frac{5}{\sqrt{10v+1}}$ for $1 < t \leq 3$. AG

[4]

4. (a)

$v(t) = \int a(t)\mathrm{d}t$ M1

$v(t) = \int 38\ln 3 \times 3^{-0.38t}\,\mathrm{d}t$

$v(t) = 38\ln 3\left(\dfrac{3^{-0.38t}}{-0.38\ln 3}\right) + C$ A1

$v(t) = -100 \times 3^{-0.38t} + C$ A1

$0 = -100 \times 3^{-0.38(0)} + C$ M1

$0 = -100 + C$

$C = 100$

$s(t) = \int v(t)\mathrm{d}t$ M1

$s(t) = \int (-100 \times 3^{-0.38t} + 100)\mathrm{d}t$

$s(t) = -100\left(\dfrac{3^{-0.38t}}{-0.38\ln 3}\right) + 100t + D$ A1

$s(t) = \dfrac{5000}{19\ln 3} \times 3^{-0.38t} + 100t + D$ A1

$\dfrac{5000}{19\ln 3} = \dfrac{5000}{19\ln 3} \times 3^{-0.38(0)} + 100(0) + D$ M1

$\dfrac{5000}{19\ln 3} = \dfrac{5000}{19\ln 3} + D$

$D = 0$

$\therefore s(t) = \dfrac{5000}{19\ln 3} \times 3^{-0.38t} + 100t$ A1

$s(t) = \dfrac{5000}{19\ln 3}\left(3^{-0.38t} + \dfrac{19\ln 3}{50}t\right)$

$s(t) = \dfrac{5000}{19\ln 3}(3^{-0.38t} + (0.38\ln 3)t)$ AG

[9]

(b)

$s'(t) = a(t)$

$v(t) = a(t)$ (M1) for setting equation

$-100 \times 3^{-0.38t} + 100 = 38\ln 3 \times 3^{-0.38t}$

$(38\ln 3 + 100) \times 3^{-0.38t} - 100 = 0$

By considering the graph of

$y = (38\ln 3 + 100) \times 3^{-0.38t} - 100$, $t = 0.8356846$. (M1) for valid approach

$\therefore t = 0.836$ A1

[3]

(c) The total distance travelled

$= \int_6^{10} |v(t)|\, dt$ (M1) for valid approach

$= \int_6^{10} \left|-100 \times 3^{-0.38t} + 100\right| dt$ (A1) for substitution

$= 384.1164218 \text{ m}$

$= 384 \text{ m}$ A1

[3]

(d) $v = -100 \times 3^{-0.38t} + 100$

$v = 100(1 - 3^{-0.38t})$

$0.01v = 1 - 3^{-0.38t}$

$3^{-0.38t} = 1 - 0.01v$ A1

$\frac{dv}{dt} = 38 \ln 3 \times 3^{-0.38t}$ A1

$\frac{dv}{dt} = 38 \ln 3(1 - 0.01v)$ M1

$\frac{dv}{dt} = 38 \ln 3\left(\frac{100 - v}{100}\right)$

$\therefore \frac{dt}{dv} = \frac{50}{19 \ln 3(100 - v)}$ AG

[3]

Chapter 14 Solution

Exercise 62

1. $$\lim_{x\to 0}\frac{\sin\frac{\pi}{4}x}{\ln(x+1)}=\lim_{x\to 0}\frac{\left(\cos\frac{\pi}{4}x\right)\left(\frac{\pi}{4}\right)}{\left(\frac{1}{x+1}\right)(1)}\left(\because\frac{0}{0}\right)$$ M1A2

$$\lim_{x\to 0}\frac{\sin\frac{\pi}{4}x}{\ln(x+1)}=\lim_{x\to 0}\frac{\pi}{4}(x+1)\cos\frac{\pi}{4}x$$

$$\lim_{x\to 0}\frac{\sin\frac{\pi}{4}x}{\ln(x+1)}=\frac{\pi}{4}(0+1)\cos\frac{\pi}{4}(0)$$ M1

$$\lim_{x\to 0}\frac{\sin\frac{\pi}{4}x}{\ln(x+1)}=\frac{\pi}{4}$$ A1

[5]

2. $$\lim_{x\to 0}\frac{3^x-1}{\sin 2x}=\lim_{x\to 0}\frac{3^x\ln 3-0}{(\cos 2x)(2)}\left(\because\frac{0}{0}\right)$$ M1A2

$$\lim_{x\to 0}\frac{3^x-1}{\sin 2x}=\lim_{x\to 0}\frac{3^x\ln 3}{2\cos 2x}$$

$$\lim_{x\to 0}\frac{3^x-1}{\sin 2x}=\frac{3^0\ln 3}{2\cos 2(0)}$$ M1

$$\lim_{x\to 0}\frac{3^x-1}{\sin 2x}=\frac{\ln 3}{2}$$ A1

[5]

3. $\lim_{x\to 0}\frac{x^2}{\cos 2x-\cos x}=\lim_{x\to 0}\frac{2x}{(-\sin 2x)(2)-(-\sin x)}\left(\because \frac{0}{0}\right)$ M1A2

$\lim_{x\to 0}\frac{x^2}{\cos 2x-\cos x}=\lim_{x\to 0}\frac{2x}{-2\sin 2x+\sin x}$

$\lim_{x\to 0}\frac{x^2}{\cos 2x-\cos x}=\lim_{x\to 0}\frac{2(1)}{-2(\cos 2x)(2)+\cos x}\left(\because \frac{0}{0}\right)$ A2

$\lim_{x\to 0}\frac{x^2}{\cos 2x-\cos x}=\lim_{x\to 0}\frac{2}{-4\cos 2x+\cos x}$

$\lim_{x\to 0}\frac{x^2}{\cos 2x-\cos x}=\frac{2}{-4\cos 2(0)+\cos 0}$ M1

$\lim_{x\to 0}\frac{x^2}{\cos 2x-\cos x}=\frac{2}{-4+1}$

$\lim_{x\to 0}\frac{x^2}{\cos 2x-\cos x}=-\frac{2}{3}$ A1

[7]

4. $\lim_{x\to 0}\frac{1-\sec x}{\sin 2x+\cos 2x-1-2x}$

$=\lim_{x\to 0}\frac{0-\sec x\tan x}{(\cos 2x)(2)+(-\sin 2x)(2)-0-2(1)}\left(\because \frac{0}{0}\right)$ M1A2

$=\lim_{x\to 0}\frac{-\sec x\tan x}{2\cos 2x-2\sin 2x-2}$

$=\lim_{x\to 0}\frac{(-\sec x\tan x)(\tan x)+(-\sec x)(\sec^2 x)}{2(-\sin 2x)(2)-2(\cos 2x)(2)-0}\left(\because \frac{0}{0}\right)$ A2

$=\lim_{x\to 0}\frac{-\sec x\tan^2 x-\sec^3 x}{-4\sin 2x-4\cos 2x}$

$=\frac{-\sec 0\tan^2 0-\sec^3 0}{-4\sin 2(0)-4\cos 2(0)}$ M1

$=\frac{0-1}{0-4}$

$=\frac{1}{4}$ A1

[7]

Exercise 63

1. $f(0) = e^{2(0)} \tan 0 = 0$ (A1) for correct value

$f'(x) = 2e^{2x} \tan x + e^{2x} \sec^2 x$

$f'(x) = e^{2x}(2\tan x + \sec^2 x)$

$f'(0) = e^{2(0)}(2\tan 0 + \sec^2 0) = 1$ (A1) for correct value

$f''(x) = 2e^{2x}(2\tan x + \sec^2 x)$
$+ e^{2x}(2\sec^2 x + 2\sec x(\sec x \tan x))$ (M1) for valid approach

$f''(x) = e^{2x}(4\tan x + 4\sec^2 x + 2\sec^2 x \tan x)$

$f''(0) = e^{2(0)}(4\tan 0 + 4\sec^2 0 + 2\sec^2 0 \tan 0) = 4$ (A1) for correct value

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!} f''(0) + \cdots$$

$f(x) = 0 + x(1) + \frac{x^2}{2}(4) + \cdots$ M1

$f(x) = x + 2x^2 + \cdots$ A1

[6]

2. $f(0)=(1+0)^{\frac{2}{3}}=1$ (A1) for correct value

$f'(x)=\frac{2}{3}(1+x)^{-\frac{1}{3}}(1)$

$f'(x)=\frac{2}{3}(1+x)^{-\frac{1}{3}}$

$f'(0)=\frac{2}{3}(1+0)^{-\frac{1}{3}}=\frac{2}{3}$ (A1) for correct value

$f''(x)=\frac{2}{3}\left(-\frac{1}{3}\right)(1+x)^{-\frac{4}{3}}(1)$

$f''(x)=-\frac{2}{9}(1+x)^{-\frac{4}{3}}$

$f''(0)=-\frac{2}{9}(1+0)^{-\frac{4}{3}}=-\frac{2}{9}$ (A1) for correct value

$f^{(3)}(x)=-\frac{2}{9}\left(-\frac{4}{3}\right)(1+x)^{-\frac{7}{3}}(1)$

$f^{(3)}(x)=\frac{8}{27}(1+x)^{-\frac{7}{3}}$

$f^{(3)}(0)=\frac{8}{27}(1+0)^{-\frac{7}{3}}=\frac{8}{27}$ (A1) for correct value

$f(x)=f(0)+xf'(0)+\frac{x^2}{2!}f''(0)+\frac{x^3}{3!}f^{(3)}(0)+\cdots$

$f(x)=1+x\left(\frac{2}{3}\right)+\frac{x^2}{2}\left(-\frac{2}{9}\right)+\frac{x^3}{6}\left(\frac{8}{27}\right)+\cdots$ M1

$f(x)=1+\frac{2}{3}x-\frac{1}{9}x^2+\frac{4}{81}x^3+\cdots$ A1

[6]

3. $f(0)=3(0)^2-\pi^2(0)+\frac{\pi^2}{2}-\pi\arccos\pi(0)=0$ (A1) for correct approach

$f'(x)=6x-\pi^2+0-\pi\left(-\frac{\pi}{\sqrt{1-(\pi x)^2}}\right)$ (M1) for valid approach

$f'(x)=6x-\pi^2+\frac{\pi^2}{\sqrt{1-\pi^2x^2}}$

$f'(0)=6(0)-\pi^2+\frac{\pi^2}{\sqrt{1-\pi^2(0)^2}}=0$ (A1) for correct approach

$f''(x)=6-0-\frac{\pi^2}{2}(1-\pi^2x^2)^{-\frac{3}{2}}(-2\pi^2x)$ (M1) for valid approach

$f''(x)=6+\pi^4x(1-\pi^2x^2)^{-\frac{3}{2}}$

$f''(0)=6+\pi^4(0)(1-\pi^2 0^2)^{-\frac{3}{2}}=6$ (A1) for correct value

The first non-zero term

$=\frac{x^2}{2!}(6)$ M1

$=3x^2$ A1

[7]

4. $f(0)=0+\cos 0-1=0$ (A1) for correct approach

$f'(x)=3x^2-(\sin(x^3))(3x^2)-0$ (M1) for valid approach

$f'(x)=3x^2-3x^2\sin(x^3)$

$f'(0)=3(0)^2-3(0)^2\sin 0=0$ (A1) for correct approach

$f''(x)=6x-((6x)(\sin(x^3))+3x^2(\cos(x^3))(3x^2))$

$f''(x)=6x-6x\sin(x^3)-9x^4\cos(x^3)$

$f''(0)=6(0)-6(0)\sin 0-9(0)^4\cos 0=0$ (A1) for correct approach

$f^{(3)}(x)=6-((6)(\sin(x^3))+6x(\cos(x^3))(3x^2))$

$-((36x^3)(\cos(x^3))+9x^4(-\sin(x^3))(3x^2))$

$f^{(3)}(x)=6+(27x^6-6)\sin(x^3)-54x^3\cos(x^3)$

$f^{(3)}(0)=6+(27(0)^6-6)\sin 0-54(0)^3\cos 0=6$ (A1) for correct value

The first non-zero term

$=\frac{x^3}{3!}(6)$ M1

$=x^3$ A1

[7]

Exercise 64

1. (a)

$f(0) = e^{3(0)} = 1$ (A1) for correct value

$f'(x) = 3e^{3x}$

$f'(0) = 3e^{3(0)} = 3$ (A1) for correct value

$f''(x) = 3(3e^{3x})$

$f''(x) = 9e^{3x}$

$f''(0) = 9e^{3(0)} = 9$ (A1) for correct value

$f^{(3)}(x) = 9(3e^{3x})$

$f^{(3)}(x) = 27e^{3x}$

$f^{(3)}(0) = 27e^{3(0)} = 27$ (A1) for correct value

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \frac{x^3}{3!}f^{(3)}(0) + \cdots$$

$f(x) = 1 + x(3) + \frac{x^2}{2}(9) + \frac{x^3}{6}(27) + \cdots$ M1

$f(x) = 1 + 3x + \frac{9}{2}x^2 + \frac{9}{2}x^3 + \cdots$ A1

[6]

(b)

$3x = 0.03$

$x = 0.01$ (A1) for correct value

$\therefore e^{0.03} \approx 1 + 3(0.01) + \frac{9}{2}(0.01)^2 + \frac{9}{2}(0.01)^3$ M1

$e^{0.03} \approx 1.0304545$ A1

[3]

2. (a) $f(0)=\ln\tan\left(0+\frac{\pi}{4}\right)=0$ (A1) for correct value

$$f'(x)=\left(\frac{1}{\tan\left(x+\frac{\pi}{4}\right)}\right)\left(\sec^2\left(x+\frac{\pi}{4}\right)\right)$$

$$f'(x)=\frac{1}{\sin\left(x+\frac{\pi}{4}\right)\cos\left(x+\frac{\pi}{4}\right)}$$

$$f'(x)=\frac{2}{\sin\left(2x+\frac{\pi}{2}\right)}$$

$$f'(0)=\frac{2}{\sin\left(2(0)+\frac{\pi}{2}\right)}=2$$ (A1) for correct value

$$f''(x)=2(-1)\left(\frac{1}{\sin^2\left(2x+\frac{\pi}{2}\right)}\right)\left(\cos\left(2x+\frac{\pi}{2}\right)\right)(2)$$ (M1) for valid approach

$$f''(x)=-\frac{4\cos\left(2x+\frac{\pi}{2}\right)}{\sin^2\left(2x+\frac{\pi}{2}\right)}$$

$$f''(0)=-\frac{4\cos\left(2(0)+\frac{\pi}{2}\right)}{\sin^2\left(2(0)+\frac{\pi}{2}\right)}=0$$ (A1) for correct value

$$f(x)=f(0)+xf'(0)+\frac{x^2}{2!}f''(0)+\cdots$$

$$f(x)=0+x(2)+\frac{x^2}{2}(0)+\cdots$$ M1

$$f(x)=2x+\cdots$$ A1

[6]

(b) $\ln\tan\left(\frac{\pi}{12}+\frac{\pi}{4}\right)\approx 2\left(\frac{\pi}{12}\right)$ M1

$\ln\tan\frac{\pi}{3}\approx\frac{\pi}{6}$

$\ln\sqrt{3}\approx\frac{\pi}{6}$ A1

$4\ln\sqrt{3}\approx 4\left(\frac{\pi}{6}\right)$

$\ln(\sqrt{3})^4\approx\frac{2\pi}{3}$

$\ln 9\approx\frac{2\pi}{3}$ A1

[3]

3. (a) $f(0) = \frac{1}{5}\ln(1+0^2) = 0$ (A1) for correct value

$$f'(x) = \frac{1}{5}\left(\frac{1}{1+x^2}\right)(2x)$$

$$f'(x) = \frac{2x}{5(1+x^2)}$$

$f'(0) = \frac{2(0)}{5(1+0^2)} = 0$ (A1) for correct value

$f''(x) = \frac{2}{5}\left(\frac{(1+x^2)(1)-(x)(2x)}{(1+x^2)^2}\right)$ (M1) for valid approach

$$f''(x) = \frac{2(1-x^2)}{5(1+x^2)^2}$$

$f''(0) = \frac{2(1-0^2)}{5(1+0^2)^2} = \frac{2}{5}$ (A1) for correct value

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \cdots$$

$f(x) = 0 + x(0) + \frac{x^2}{2}\left(\frac{2}{5}\right) + \cdots$ M1

$f(x) = \frac{1}{5}x^2 + \cdots$ A1

[6]

(b) $f(x) \approx \frac{1}{5}x^2$

$f'(x) \approx \frac{1}{5}(2x)$ M1

$f'(x) \approx \frac{2}{5}x$ A1

$$\therefore f'(0.1) \approx \frac{2}{5}(0.1)$$

$f'(0.1) \approx 0.04$ A1

[3]

4. (a)

$f(0) = \cot\left(\frac{\pi}{2}(0+1)\right) = 0$ (A1) for correct value

$f'(x) = -\text{cosec}^2\left(\frac{\pi}{2}(x+1)\right)\cdot\frac{\pi}{2}$

$f'(x) = -\frac{\pi}{2}\text{cosec}^2\left(\frac{\pi}{2}(x+1)\right)$

$f'(0) = -\frac{\pi}{2}\text{cosec}^2\left(\frac{\pi}{2}(0+1)\right) = -\frac{\pi}{2}$ (A1) for correct value

$f''(x) = -\frac{\pi}{2}\left(2\text{cosec}\left(\frac{\pi}{2}(x+1)\right)\right)$

(M1) for valid approach

$\cdot\left(-\text{cosec}\left(\frac{\pi}{2}(x+1)\right)\cot\left(\frac{\pi}{2}(x+1)\right)\right)\cdot\frac{\pi}{2}$

$f''(x) = \frac{\pi^2}{2}\text{cosec}^2\left(\frac{\pi}{2}(x+1)\right)\cot\left(\frac{\pi}{2}(x+1)\right)$

$f''(0) = \frac{\pi^2}{2}\text{cosec}^2\left(\frac{\pi}{2}(0+1)\right)\cot\left(\frac{\pi}{2}(0+1)\right) = 0$ (A1) for correct value

$f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \cdots$

$f(x) = 0 + x\left(-\frac{\pi}{2}\right) + \frac{x^2}{2}(0) + \cdots$ M1

$f(x) = -\frac{\pi}{2}x + \cdots$ A1

[6]

(b)

$\int_1^{1.5} f(t)\mathrm{d}t \approx \int_1^{1.5} -\frac{\pi}{2}t\mathrm{d}t$ M1

$\int_1^{1.5} f(t)\mathrm{d}t \approx -0.9817477042$

$\int_1^{1.5} f(t)\mathrm{d}t \approx -0.982$ A1

[2]

Exercise 65

1. (a) (i)

$f'(x) = (2^{\tan x} \ln 2)(\sec^2 x)$ — (M1) for valid approach

$f'(x) = \ln 2 \cdot 2^{\tan x} \sec^2 x$ — A1

$f''(x) = \ln 2 \begin{pmatrix} (2^{\tan x} \ln 2)(\sec^2 x) \\ +(2^{\tan x})(2\sec x)(\sec x \tan x) \end{pmatrix}$ — (M1) for valid approach

$f''(x) = \ln 2 \cdot 2^{\tan x} \sec^2 x(\ln 2 + 2\tan x)$ — A1

(ii)

$f(x) = f(0) + xf'(0) + \frac{x^2}{2!} f''(0) + \cdots$

$f(x) = 2^{\tan 0} + x(\ln 2 \cdot 2^{\tan 0} \sec^2 0)$
$+ \frac{x^2}{2}(\ln 2 \cdot 2^{\tan 0} \sec^2 0(\ln 2 + 2\tan 0)) + \cdots$ — M2

$f(x) = 1 + x(\ln 2) + \frac{x^2}{2}(\ln 2 \cdot (\ln 2 + 0)) + \cdots$ — A1

$f(x) = 1 + (\ln 2)x + \frac{(\ln 2)^2}{2} x^2 + \cdots$ — A1

[8]

(b)

$e^{2x} = 1 + 2x + \frac{(2x)^2}{2!} + \cdots$ — (M1) for substitution

$e^{2x} = 1 + 2x + 2x^2 + \cdots$ — (A1) for correct values

$\ln(1 + 7e^{2x} + 12e^{4x})$

$= \ln(1 + 3e^{2x})(1 + 4e^{2x})$ — (M1) for valid approach

$= \ln(1 + 3e^{2x}) + \ln(1 + 4e^{2x})$ — (A1) for correct approach

$= 3e^{2x} - \frac{(3e^{2x})^2}{2} + \cdots + 4e^{2x} - \frac{(4e^{2x})^2}{2} + \cdots$ — (M1) for valid approach

$= 7e^{2x} - \frac{25(e^{2x})^2}{2} + \cdots$ — (A1) for simplification

$= 7(1 + 2x + 2x^2 + \cdots) - \frac{25(1 + 2x + 2x^2 + \cdots)^2}{2} + \cdots$ — (A1) for substitution

$= 7 + 14x + 14x^2 - \frac{25(1 + 4x + 8x^2 + \cdots)}{2} + \cdots$ — (A1) for correct approach

$= 7 + 14x + 14x^2 - \frac{25}{2} - 50x - 100x^2 + \cdots$

$= -\frac{11}{2} - 36x - 86x^2 + \cdots$ — A1

[9]

(c) $$\lim_{x \to 0} \frac{\ln(1+7e^{2x}+12e^{4x})}{f(x)}$$

$$= \lim_{x \to 0} \frac{-\frac{11}{2} - 36x - 86x^2 + \cdots}{1 + (\ln 2)x + \frac{(\ln 2)^2}{2}x^2 + \cdots}$$ A1

$$= \frac{-\frac{11}{2} - 0 - 0 + \cdots}{1 + 0 + 0 + \cdots}$$ M1

$$= -\frac{11}{2}$$ A1

[3]

2. (a) (i) $\arctan x = x - \frac{x^3}{3} + \frac{x^5}{5} + \cdots$

$\frac{d}{dx}(\arctan x) = \frac{d}{dx}\left(x - \frac{x^3}{3} + \frac{x^5}{5} + \cdots\right)$ (M1) for valid approach

$\frac{1}{1+x^2} = 1 - \frac{1}{3}(3x^2) + \frac{1}{5}(5x^4) + \cdots$ (A2) for correct approach

$\frac{1}{1+x^2} = 1 - x^2 + x^4 + \cdots$ A1

(ii) $g(x) = \frac{1}{1+(x^{1.5})^2}$

$g(x) = 1 - (x^{1.5})^2 + (x^{1.5})^4 + \cdots$ (A1) for substitution

$g(x) = 1 - x^3 + x^6 + \cdots$ A1

[6]

(b) $g(0.1) = 1 - 0.1^3 + 0.1^6 + \cdots$ M1

$g(0.1) > 1 - 0.1^3$

$g(0.1) > 0.999$ A1

$g(0.1) < 1 - 0.1^3 + 0.1^6$

$g(0.1) < 0.999001$ A1

$\therefore 0.999 < g(0.1) < 0.999001$ M1

$0.999 < \frac{1}{1+0.1^3} < 0.999001$

$0.999 < \frac{1}{1.001} < 0.999001$ AG

[4]

(c) $\lim_{x\to 0} \frac{g(x)-1}{e^{x^3}-1}$

$= \lim_{x\to 0} \frac{1 - x^3 + x^6 + \cdots - 1}{1 + x^3 + \frac{(x^3)^2}{2!} + \cdots - 1}$ M1A1

$= \lim_{x\to 0} \frac{-x^3 + x^6 + \cdots}{x^3 + \frac{1}{2}x^6 + \cdots}$ M1

$= \lim_{x\to 0} \frac{-1 + x^3 + \cdots}{1 + \frac{1}{2}x^3 + \cdots}$ M1A1

$= \frac{-1+0+\cdots}{1+0+\cdots}$ M1

$= -1$ AG

[6]

3. (a) $\sin x = x - \frac{x^3}{3!} + \cdots$

$\sin 2x = 2x - \frac{(2x)^3}{3!} + \cdots$ M1

$\sin 2x = 2x - \frac{4}{3}x^3 + \cdots$ A1

$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots$

$\cos 2x = 1 - \frac{(2x)^2}{2!} + \frac{(2x)^4}{4!} + \cdots$ M1

$\cos 2x = 1 - 2x^2 + \frac{2}{3}x^4 + \cdots$ A1

$\sin^2 4x = (2\sin 2x \cos 2x)^2$ A1

$\sin^2 4x = \left(2\left(2x - \frac{4}{3}x^3 + \cdots\right)\left(1 - 2x^2 + \frac{2}{3}x^4 + \cdots\right)\right)^2$ M1

$\sin^2 4x = 4\left(2x - 4x^3 - \frac{4}{3}x^3 + \cdots\right)^2$

$\sin^2 4x = 4\left(2x - \frac{16}{3}x^3 + \cdots\right)^2$ A1

$\sin^2 4x = 4\left(4x^2 - \frac{64}{3}x^4 + \cdots\right)$

$\sin^2 4x = 16x^2 - \frac{256}{3}x^4 + \cdots$ AG

[7]

(b) $\sin^2 4\left(\frac{\pi}{16}\right) > 16\left(\frac{\pi}{16}\right)^2 - \frac{256}{3}\left(\frac{\pi}{16}\right)^4$ M1A1

$\sin^2 \frac{\pi}{4} > \frac{\pi^2}{16} - \frac{\pi^4}{768}$ A1

$\sin^2 \frac{\pi}{4} > \frac{48\pi^2}{768} - \frac{\pi^4}{768}$

$\sin^2 \frac{\pi}{4} > \frac{\pi^2(48-\pi^2)}{768}$ A1

$\sin^2 4\left(\frac{\pi}{16}\right) < 16\left(\frac{\pi}{16}\right)^2$ M1A1

$\sin^2 \frac{\pi}{4} < \frac{\pi^2}{16}$ A1

$\therefore \frac{\pi^2(48-\pi^2)}{768} < \sin^2 \frac{\pi}{4} < \frac{\pi^2}{16}$ A1

$\sqrt{\frac{\pi^2(48-\pi^2)}{768}} < \sin \frac{\pi}{4} < \sqrt{\frac{\pi^2}{16}}$ M1

$\frac{\pi}{16}\sqrt{\frac{48-\pi^2}{3}} < \sin \frac{\pi}{4} < \frac{\pi}{4}$ AG

[9]

4. (a) (i)

$f'(x) = \sec x \tan x$ A1

$f''(x) = (\sec x \tan x)(\tan x) + (\sec x)(\sec^2 x)$ (M1) for valid approach

$f''(x) = \sec x \tan^2 x + \sec^3 x$

$f''(x) = \sec x(\sec^2 x - 1) + \sec^3 x$

$f''(x) = \sec^3 x - \sec x + \sec^3 x$

$f''(x) = 2\sec^3 x - \sec x$ A1

$f^{(3)}(x) = 2(3)(\sec^2 x)(\sec x \tan x)$
$- \sec x \tan x$ (M1) for valid approach

$f^{(3)}(x) = \sec x \tan x(6\sec^2 x - 1)$ A1

(ii)

$f(x) = f(0) + xf'(0)$
$+ \dfrac{x^2}{2!} f''(0) + \dfrac{x^3}{3!} f^{(3)}(0) + \cdots$

$f(x) = \sec 0 + x(\sec 0 \tan 0)$
$+ \dfrac{x^2}{2}(2\sec^3 0 - \sec 0)$ M2
$+ \dfrac{x^3}{6}(\sec 0 \tan 0(6\sec^2 0 - 1)) + \cdots$

$f(x) = 1 + x(0) + \dfrac{x^2}{2}(1) + \dfrac{x^3}{6}(0) + \cdots$ A1

$f(x) = 1 + \dfrac{1}{2}x^2 + \cdots$ A1

[9]

(b)

$\sin x = x - \dfrac{x^3}{3!} + \cdots$

$\tan x = \dfrac{\sin x}{\cos x}$

$\tan x = \sin x \sec x$ (A1) for correct approach

$\tan x = \left(x - \dfrac{x^3}{6} + \cdots\right)\left(1 + \dfrac{1}{2}x^2 + \cdots\right)$ M1A1

$\tan x = x + \dfrac{1}{2}x^3 - \dfrac{1}{6}x^3 + \cdots$ (M1) for valid approach

$\tan x = x + \dfrac{1}{3}x^3 + \cdots$ A1

[5]

(c) The approximate value of the area

$= \int_{0.5}^{1} y \tan y \, dy$ (M1) for valid approach

$\approx \int_{0.5}^{1} y\left(y + \frac{1}{3}y^3\right) dy$ (A1) for substitution

$\approx \int_{0.5}^{1} \left(y^2 + \frac{1}{3}y^4\right) dy$ (M1) for valid approach

≈ 0.35625 A1

[4]

Chapter 15 Solution

Exercise 66

1. (a) $\frac{dx}{dt}=\pi^2 x\sin \pi t$

$\frac{1}{x}dx=\pi^2 \sin \pi t dt$ (M1) for valid approach

$\int \frac{1}{x}dx=\int \pi^2 \sin \pi t dt$ (A1) for correct approach

Let $u=\pi t$. (M1) for substitution

$\frac{du}{dt}=\pi \Rightarrow du=\pi dt$

$\therefore \int \frac{1}{x}dx=\int \pi \sin u du$ (A1) for correct working

$\ln|x|=-\pi\cos u+C$

$\ln|x|=-\pi\cos \pi t+C$

$x=e^{-\pi\cos \pi t+C}$ A1

$1=e^{-\pi\cos \pi(0.5)+C}$ (M1) for substitution

$1=e^{C}$

$C=0$ (A1) for correct value

$\therefore x=e^{-\pi\cos \pi t}$ A1

[8]

(b) $e^{-\pi}\le x\le e^{\pi}$ A2

[2]

2. $\frac{dy}{dx} = \frac{B}{y^2}$

$y^2 dy = B dx$	(M1) for valid approach
$\int y^2 dy = \int B dx$	(A1) for correct approach
$\frac{1}{3} y^3 = Bx + C$	A1
$y^3 = 3Bx + C$	
$y = (3Bx + C)^{\frac{1}{3}}$	A1
$3 = (3B(4) + C)^{\frac{1}{3}}$	(M1) for substitution
$27 = 12B + C$	
$C = 27 - 12B$	
$9 = (3B(121) + C)^{\frac{1}{3}}$	
$\therefore 729 = 363B + 27 - 12B$	(M1) for substitution
$702 = 351B$	
$B = 2$	
$C = 27 - 12(2)$	
$C = 3$	(A1) for correct value
$\therefore y = \left(3(2)\left(\frac{61}{3}\right) + 3\right)^{\frac{1}{3}}$	
$y = 125^{\frac{1}{3}}$	
$y = 5$	A1

[8]

3. $\frac{dy}{dx} = -y^3 \sin^2 x$

$-\frac{1}{y^3} dy = \sin^2 x dx$ (M1) for valid approach

$\int -\frac{1}{y^3} dy = \int \sin^2 x dx$ (A1) for correct approach

$\int -\frac{1}{y^3} dy = \int \frac{1-\cos 2x}{2} dx$ (A1) for correct approach

$\int -\frac{1}{y^3} dy = \frac{1}{2}\int (1-\cos 2x) dx$

$\frac{1}{2y^2} = \frac{1}{2}\left(x - \frac{1}{2}\sin 2x\right) + C$ A1

$\frac{1}{y^2} = x - \frac{1}{2}\sin 2x + C$

$y = \frac{1}{\sqrt{x - \frac{1}{2}\sin 2x + C}}$ A1

$2 = \frac{1}{\sqrt{0 - \frac{1}{2}\sin 2(0) + C}}$ (M1) for substitution

$2 = \frac{1}{\sqrt{C}}$

$\sqrt{C} = \frac{1}{2}$

$C = \frac{1}{4}$ (A1) for correct value

$\therefore y = \frac{1}{\sqrt{x - \frac{1}{2}\sin 2x + \frac{1}{4}}}$

$y = \frac{1}{\sqrt{\frac{1}{4}(4x - 2\sin 2x + 1)}}$

$y = \frac{2}{\sqrt{4x - 2\sin 2x + 1}}$ A1

[8]

4. $\dfrac{dy}{dx} = \dfrac{y}{(2x+3)\ln y}$

$\dfrac{\ln y}{y}dy = \dfrac{1}{2x+3}dx$ (M1) for valid approach

$\int \dfrac{\ln y}{y}dy = \int \dfrac{1}{2x+3}dx$ (A1) for correct approach

Let $u = \ln y$. (M1) for substitution

$\dfrac{du}{dy} = \dfrac{1}{y} \Rightarrow du = \dfrac{1}{y}dy$

$\therefore \int u du = \int \dfrac{1}{2x+3}dx$ (A1) for correct working

$\dfrac{1}{2}u^2 = \dfrac{1}{2}\ln|2x+3| + C$

$(\ln y)^2 = \ln|2x+3| + C$ (M1) for valid approach

$\ln y = \sqrt{\ln|2x+3| + C}$

$y = e^{\sqrt{\ln|2x+3|+C}}$ A1

$e^{\sqrt{\ln 5}} = e^{\sqrt{\ln|2(1)+3|+C}}$ (M1) for substitution

$e^{\sqrt{\ln 5}} = e^{\sqrt{\ln 5+C}}$

$\sqrt{\ln 5} = \sqrt{\ln 5 + C}$

$C = 0$ (A1) for correct value

$\therefore y = e^{\sqrt{\ln|2x+3|}}$ A1

[9]

Exercise 67

1. $\frac{dy}{dx} - \frac{y}{x} = \frac{1}{x^6}$

$\frac{dy}{dx} - \frac{1}{x}y = \frac{1}{x^6}$

The integrating factor

$= e^{\int -\frac{1}{x}dx}$ (M1) for valid approach

$= e^{-\ln x}$

$= e^{\ln x^{-1}}$

$= \frac{1}{x}$ A1

$\therefore \frac{1}{x}\frac{dy}{dx} - \frac{1}{x}\cdot\frac{1}{x}y = \frac{1}{x}\cdot\frac{1}{x^6}$ (M1) for valid approach

$\frac{1}{x}\frac{dy}{dx} - \frac{1}{x^2}y = \frac{1}{x^7}$

$\frac{d}{dx}\left(\frac{y}{x}\right) = \frac{1}{x^7}$ (A1) for correct approach

$\frac{y}{x} = \int \frac{1}{x^7}dx$

$\frac{y}{x} = -\frac{1}{6x^6} + C$

$y = -\frac{1}{6x^5} + Cx$ A1

$\frac{11}{6} = -\frac{1}{6(1)^5} + C(1)$ (M1) for substitution

$C = 2$ (A1) for correct value

$\therefore y = -\frac{1}{6x^5} + 2x$ A1

[8]

2. $x\dfrac{dy}{dx} - y = x^2 2^x$

$\dfrac{dy}{dx} - \dfrac{1}{x}y = x2^x$ (A1) for correct approach

The integrating factor

$= e^{\int -\frac{1}{x}dx}$ (M1) for valid approach

$= e^{-\ln x}$

$= e^{\ln x^{-1}}$

$= \dfrac{1}{x}$ A1

$\therefore \dfrac{1}{x}\dfrac{dy}{dx} - \dfrac{1}{x}\cdot\dfrac{1}{x}y = \dfrac{1}{x}\cdot x2^x$ (M1) for valid approach

$\dfrac{1}{x}\dfrac{dy}{dx} - \dfrac{1}{x^2}y = 2^x$

$\dfrac{d}{dx}\left(\dfrac{y}{x}\right) = 2^x$ (A1) for correct approach

$\dfrac{y}{x} = \int 2^x dx$

$\dfrac{y}{x} = \dfrac{1}{\ln 2}2^x + C$

$y = \dfrac{1}{\ln 2}x2^x + Cx$ A1

$\dfrac{3}{\ln 2} = \dfrac{1}{\ln 2}(1)2^1 + C(1)$ (M1) for substitution

$C = \dfrac{1}{\ln 2}$ (A1) for correct value

$\therefore y = \dfrac{1}{\ln 2}x2^x + \dfrac{1}{\ln 2}x$

$y = \dfrac{1}{\ln 2}x(2^x + 1)$ A1

[9]

3. $\frac{dy}{dx} = \cos^3 x - 2y\tan x$

$\frac{dy}{dx} + (2\tan x)y = \cos^3 x$

Let $u = \cos x$. (M1) for substitution

$\frac{du}{dx} = -\sin x \Rightarrow (-1)\cdot du = \sin x dx$

The integrating factor

$= e^{\int 2\tan x dx}$ (M1) for valid approach

$= e^{\int \frac{2\sin x}{\cos x} dx}$

$= e^{\int -\frac{2}{u} du}$

$= e^{-2\ln u}$

$= e^{\ln u^{-2}}$

$= \frac{1}{u^2}$

$= \frac{1}{\cos^2 x}$ A1

$\therefore \frac{1}{\cos^2 x}\frac{dy}{dx} + \frac{1}{\cos^2 x}(2\tan x)y = \frac{1}{\cos^2 x}\cdot\cos^3 x$ (M1) for valid approach

$\frac{1}{\cos^2 x}\frac{dy}{dx} + \frac{2\sin x}{\cos^3 x}\cdot y = \cos x$

$\frac{d}{dx}\left(\frac{y}{\cos^2 x}\right) = \cos x$ (A1) for correct approach

$\frac{y}{\cos^2 x} = \int \cos x dx$

$\frac{y}{\cos^2 x} = \sin x + C$

$y = \cos^2 x(\sin x + C)$ A1

$3 = \cos^2 0(\sin 0 + C)$ (M1) for substitution

$3 = 0 + C$

$C = 3$ (A1) for correct value

$\therefore y = \cos^2 x(\sin x + 3)$ A1

[9]

4. $\sin x\frac{dy}{dx}+\cos^2 x=1-y\cos x$

$\sin x\frac{dy}{dx}+y\cos x=1-\cos^2 x$

$\sin x\frac{dy}{dx}+y\cos x=\sin^2 x$ A1

$\frac{dy}{dx}+(\cot x)y=\sin x$ A1

Let $u=\sin x$. M1

$\frac{du}{dx}=\cos x \Rightarrow du=\cos x dx$

The integrating factor

$=e^{\int \cot x dx}$ M1

$=e^{\int \frac{\cos x}{\sin x}dx}$

$=e^{\int \frac{1}{u}du}$

$=e^{\ln u}$

$=u$

$=\sin x$ A1

$\therefore \sin x\frac{dy}{dx}+\sin x\cdot(\cot x)y=\sin x\cdot\sin x$ M1

$\sin x\frac{dy}{dx}+(\cos x)y=\sin^2 x$

$\frac{d}{dx}(y\sin x)=\sin^2 x$ A1

$y\sin x=\int \sin^2 x dx$

$y\sin x=\int \frac{1}{2}(1-\cos 2x)dx$ A1

$y\sin x=\int\left(\frac{1}{2}-\frac{1}{2}\cos 2x\right)dx$

$y\sin x=\frac{1}{2}x-\frac{1}{4}\sin 2x+C$

$y\sin x=\frac{2x-\sin 2x+C}{4}$

$y=\frac{2x-\sin 2x+C}{4\sin x}$ A1

$\pi=\frac{2\left(\frac{\pi}{2}\right)-\sin 2\left(\frac{\pi}{2}\right)+C}{4\sin\frac{\pi}{2}}$ M1

$$\pi = \frac{\pi + C}{4}$$

$$4\pi = \pi + C$$

$$C = 3\pi$$

$$\therefore y = \frac{2x - \sin 2x + 3\pi}{4\sin x}$$ AG

[10]

Exercise 68

1. $\begin{cases} x_{n+1} = x_n + 0.1 \\ y_{n+1} = y_n + 0.1 \left.\dfrac{\mathrm{d}y}{\mathrm{d}x}\right|_{(x_n,\, y_n)} \end{cases}$ (M1) for valid approach

$x_0 = 4,\ y_0 = 10$ (A1) for correct values

$x_1 = 4 + 0.1 = 4.1$

$y_1 = 10 + 0.1\left(\dfrac{(4)(10)}{4^2 - 9}\right) = 10.57142857$ A1

$x_2 = 4.1 + 0.1 = 4.2$

$y_2 = 10.57142857 + 0.1\left(\dfrac{(4.1)(10.57142857)}{4.1^2 - 9}\right)$

$y_2 = 11.12639473$ A1

$x_3 = 4.2 + 0.1 = 4.3$

$y_3 = 11.12639473 + 0.1\left(\dfrac{(4.2)(11.12639473)}{4.2^2 - 9}\right)$

$y_3 = 11.66726114$ A1

Thus, the required approximation is 11.7. A1

[6]

2. $\begin{cases} x_{n+1} = x_n + 0.2 \\ y_{n+1} = y_n + 0.2 \left.\dfrac{\mathrm{d}y}{\mathrm{d}x}\right|_{(x_n,\, y_n)} \end{cases}$ (M1) for valid approach

$x_0 = 0,\ y_0 = 2$ (A1) for correct values

$x_1 = 0 + 0.2 = 0.2$

$y_1 = 2 + 0.2(e^0 + 4(2)) = 3.8$ A1

$x_2 = 0.2 + 0.2 = 0.4$

$y_2 = 3.8 + 0.2(e^{0.2} + 4(3.8)) = 7.084280552$ A1

$x_3 = 0.4 + 0.2 = 0.6$

$y_3 = 7.084280552 + 0.2(e^{0.4} + 4(7.084280552))$

$y_3 = 13.05006993$ A1

$x_4 = 0.6 + 0.2 = 0.8$

$y_4 = 13.05006993 + 0.2(e^{0.6} + 4(13.05006993))$

$y_4 = 23.85454964$ A1

Thus, the required approximation is 23.9. A1

[7]

3. $\dfrac{dy}{dx} = \dfrac{y}{x} + \dfrac{1}{x^3}$

$$\begin{cases} x_{n+1} = x_n + 0.25 \\ y_{n+1} = y_n + 0.25 \left.\dfrac{dy}{dx}\right|_{(x_n,\, y_n)} \end{cases}$$ (M1) for valid approach

$x_0 = 1,\ y_0 = 2$ (A1) for correct values

$x_1 = 1 + 0.25 = 1.25$

$y_1 = 2 + 0.25\left(\dfrac{2}{1} + \dfrac{1}{1^3}\right) = 2.75$ A1

$x_2 = 1.25 + 0.25 = 1.5$

$y_2 = 2.75 + 0.25\left(\dfrac{2.75}{1.25} + \dfrac{1}{1.25^3}\right) = 3.428$ A1

$x_3 = 1.5 + 0.25 = 1.75$

$y_3 = 3.428 + 0.25\left(\dfrac{3.428}{1.5} + \dfrac{1}{1.5^3}\right) = 4.073407407$ A1

$x_4 = 1.75 + 0.25 = 2$

$y_4 = 4.073407407 + 0.25\left(\dfrac{4.073407407}{1.75} + \dfrac{1}{1.75^3}\right)$

$y_4 = 4.701969982$ A1

Thus, the required approximation is 4.70. A1

[7]

4. $\frac{1}{x}\frac{\mathrm{d}y}{\mathrm{d}x}-y=-1$

$\frac{\mathrm{d}y}{\mathrm{d}x}-xy=-x$

$\frac{\mathrm{d}y}{\mathrm{d}x}=xy-x$ (M1) for valid approach

$\begin{cases} x_{n+1}=x_n+0.2 \\ y_{n+1}=y_n+0.2\left.\frac{\mathrm{d}y}{\mathrm{d}x}\right|_{(x_n,\,y_n)} \end{cases}$ (M1) for valid approach

$x_0=3,\ y_0=3$ (A1) for correct values

$x_1=3+0.2=3.2$

$y_1=3+0.2((3)(3)-3)=4.2$ A1

$x_2=3.2+0.2=3.4$

$y_2=4.2+0.2((3.2)(4.2)-3.2)=6.248$ A1

$x_3=3.4+0.2=3.6$

$y_3=6.248+0.2((3.4)(6.248)-3.4)=9.81664$ A1

$x_4=3.6+0.2=3.8$

$y_4=9.81664+0.2((3.6)(9.81664)-3.6)=16.1646208$ A1

$x_5=3.8+0.2=4$

$y_5=16.1646208+0.2((3.8)(16.1646208)-3.8)$

$y_5=27.68973261$ A1

Thus, the required approximation is 27.7. A1

[9]

Exercise 69

1. $\dfrac{dy}{dx} - 4y = e^{2x}$

$\dfrac{dy}{dx} = 4y + e^{2x}$

$\left.\dfrac{dy}{dx}\right|_{x=0} = 4(1) + e^{2(0)} = 5$ (A1) for correct value

$\dfrac{d^2y}{dx^2} = 4\dfrac{dy}{dx} + 2e^{2x}$ (A1) for correct approach

$\left.\dfrac{d^2y}{dx^2}\right|_{x=0} = 4(5) + 2e^{2(0)} = 22$ (A1) for correct value

$y = y(0) + x\left.\dfrac{dy}{dx}\right|_{x=0} + \dfrac{x^2}{2!}\left.\dfrac{d^2y}{dx^2}\right|_{x=0} + \cdots$

$y = 1 + x(5) + \dfrac{x^2}{2}(22) + \cdots$ M1

$y = 1 + 5x + 11x^2 + \cdots$ A1

[5]

2. $\dfrac{dy}{dx} - xy = \ln(x+1)$

$\dfrac{dy}{dx} = xy + \ln(x+1)$

$\left.\dfrac{dy}{dx}\right|_{x=0} = (0)(-2) + \ln(0+1) = 0$ (A1) for correct value

$\dfrac{d^2y}{dx^2} = (1)(y) + x\dfrac{dy}{dx} + \dfrac{1}{x+1} = y + x\dfrac{dy}{dx} + \dfrac{1}{x+1}$ (A1) for correct approach

$\left.\dfrac{d^2y}{dx^2}\right|_{x=0} = -2 + (0)(0) + \dfrac{1}{0+1} = -1$ (A1) for correct value

$y = y(0) + x\left.\dfrac{dy}{dx}\right|_{x=0} + \dfrac{x^2}{2!}\left.\dfrac{d^2y}{dx^2}\right|_{x=0} + \cdots$

$y = -2 + x(0) + \dfrac{x^2}{2}(-1) + \cdots$ M1

$y = -2 - \dfrac{1}{2}x^2 + \cdots$ A1

[5]

3. $e^{2x}\dfrac{dy}{dx} - e^{x}y = 1$

$\dfrac{dy}{dx} - e^{-x}y = e^{-2x}$

$\dfrac{dy}{dx} = e^{-x}y + e^{-2x}$ (M1) for valid approach

$\left.\dfrac{dy}{dx}\right|_{x=0} = e^{-0}(e) + e^{-2(0)} = e + 1$ (A1) for correct value

$\dfrac{d^2y}{dx^2} = (-e^{-x})(y) + e^{-x}\dfrac{dy}{dx} - 2e^{2x} = -e^{-x}y + e^{-x}\dfrac{dy}{dx} - 2e^{2x}$ (A1) for correct approach

$\left.\dfrac{d^2y}{dx^2}\right|_{x=0} = -e^{-0}(e) + e^{-0}(e+1) - 2e^{2(0)} = -1$ (A1) for correct value

$y = y(0) + x\left.\dfrac{dy}{dx}\right|_{x=0} + \dfrac{x^2}{2!}\left.\dfrac{d^2y}{dx^2}\right|_{x=0} + \cdots$

$y = e + x(e+1) + \dfrac{x^2}{2}(-1) + \cdots$ M1

$y = e + (e+1)x - \dfrac{1}{2}x^2 + \cdots$ A1

[6]

4. $e^{3x}\dfrac{dy}{dx}-(3x^2+2x-1)e^{3x}y-e^{-3x}=0$

$\dfrac{dy}{dx}-(3x^2+2x-1)y-e^{-6x}=0$

$\dfrac{dy}{dx}=(3x^2+2x-1)y+e^{-6x}$ (M1) for valid approach

$\left.\dfrac{dy}{dx}\right|_{x=0}=(3(0)^2+2(0)-1)(2)+e^{-6(0)}=-1$ (A1) for correct value

$\dfrac{d^2y}{dx^2}=(6x+2)(y)+(3x^2+2x-1)\dfrac{dy}{dx}-6e^{-6x}$

$\dfrac{d^2y}{dx^2}=(6x+2)y+(3x^2+2x-1)\dfrac{dy}{dx}-6e^{-6x}$ (A1) for correct approach

$\left.\dfrac{d^2y}{dx^2}\right|_{x=0}=(6(0)+2)(2)+(3(0)^2+2(0)-1)(-1)-6e^{-6(0)}=-1$ (A1) for correct value

$\dfrac{d^3y}{dx^3}=(6)(y)+(6x+2)\dfrac{dy}{dx}$

$+(6x+2)\dfrac{dy}{dx}+(3x^2+2x-1)\dfrac{d^2y}{dx^2}-6(-6)e^{-6x}$

$\dfrac{d^3y}{dx^3}=6y+(12x+4)\dfrac{dy}{dx}+(3x^2+2x-1)\dfrac{d^2y}{dx^2}+36e^{-6x}$ (A1) for correct approach

$\left.\dfrac{d^3y}{dx^3}\right|_{x=0}=6(2)+(12(0)+4)(-1)$

$+(3(0)^2+2(0)-1)(-1)+36e^{-6(0)}$

$\left.\dfrac{d^3y}{dx^3}\right|_{x=0}=45$ (A1) for correct value

$y=y(0)+x\left.\dfrac{dy}{dx}\right|_{x=0}+\dfrac{x^2}{2!}\left.\dfrac{d^2y}{dx^2}\right|_{x=0}+\dfrac{x^3}{3!}\left.\dfrac{d^3y}{dx^3}\right|_{x=0}+\cdots$

$y=2+x(-1)+\dfrac{x^2}{2}(-1)+\dfrac{x^3}{6}(45)+\cdots$ M1

$y=2-x-\dfrac{1}{2}x^2+\dfrac{15}{2}x^3+\cdots$ A1

[8]

Exercise 70

1. (a) $\frac{dv}{dt} = \sqrt{4 - v^2}$

$\frac{1}{\sqrt{4-v^2}} dv = dt$ (M1) for valid approach

$\int \frac{1}{\sqrt{4-v^2}} dv = \int dt$ (A1) for correct approach

$\int \frac{1}{\sqrt{2^2 - v^2}} dv = \int dt$

$\arcsin \frac{v}{2} = t + C$ A1

$\frac{v}{2} = \sin(t + C)$

$v = 2\sin(t + C)$ A1

$2 = 2\sin(0 + C)$ (M1) for substitution

$1 = \sin C$

$\sin C = \sin \frac{\pi}{2}$

$C = \frac{\pi}{2}$ (A1) for correct value

$\therefore v = 2\sin\left(t + \frac{\pi}{2}\right)$ A1

[7]

(b) The total distance travelled

$= \int_0^{\frac{2\pi}{3}} |v(t)| dt$ (M1) for valid approach

$= \int_0^{\frac{2\pi}{3}} \left|2\sin\left(t + \frac{\pi}{2}\right)\right| dt$ (A1) for substitution

$= 2.267949192$ m

$= 2.27$ m A1

[3]

(c) $s = \int v \mathrm{d}t$

$s = \int 2\sin\left(t + \frac{\pi}{2}\right)\mathrm{d}t$ (M1) for valid approach

$s = -2\cos\left(t + \frac{\pi}{2}\right) + D$ A1

$0 = -2\cos\left(\frac{\pi}{2} + \frac{\pi}{2}\right) + D$ (M1) for substitution

$0 = 2 + D$

$D = -2$ (A1) for correct value

$\therefore s = -2\cos\left(t + \frac{\pi}{2}\right) - 2$ A1

[5]

(d) $s = -2\cos\left(t + \frac{\pi}{2}\right) - 2$

$s + 2 = -2\cos\left(t + \frac{\pi}{2}\right)$

$-\frac{s+2}{2} = \cos\left(t + \frac{\pi}{2}\right)$ A1

$v = 2\sin\left(t + \frac{\pi}{2}\right)$

$v = 2\sqrt{1 - \cos^2\left(t + \frac{\pi}{2}\right)}$ M1

$\therefore v = 2\sqrt{1 - \left(-\frac{s+2}{2}\right)^2}$ A1

$v = 2\sqrt{1 - \frac{s^2 + 4s + 4}{4}}$

$v = 2\sqrt{\frac{4 - s^2 - 4s - 4}{4}}$ M1

$v = \frac{2\sqrt{-s^2 - 4s}}{2}$ A1

$v = \sqrt{-s(s+4)}$ AG

[5]

2. (a)

$a=\dfrac{v+150}{300}$

$\dfrac{dv}{dt}=\dfrac{v+150}{300}$

$\dfrac{1}{v+150}dv=\dfrac{1}{300}dt$ (M1) for valid approach

$\int\dfrac{1}{v+150}dv=\int\dfrac{1}{300}dt$ (A1) for correct approach

$\ln|v+150|=\dfrac{1}{300}t+C$ A1

$v+150=e^{\frac{1}{300}t+C}$

$v=e^{\frac{1}{300}t+C}-150$

$v=e^{C}\cdot e^{\frac{1}{300}t}-150$ A1

$0=e^{C}\cdot e^{\frac{1}{300}(0)}-150$ (M1) for substitution

$e^{C}=150$ (A1) for correct value

$\therefore v=150e^{\frac{1}{300}t}-150$ A1

[7]

(b)

$\ln(v+150)=\dfrac{1}{300}t+\ln 150$

$\ln(5+150)=\dfrac{1}{300}t+\ln 150$ (M1) for setting equation

$\ln 155-\ln 150=\dfrac{1}{300}t$

$\ln\dfrac{31}{30}=\dfrac{1}{300}t$ (A1) for correct approach

$t=300\ln\dfrac{31}{30}$ s A1

[3]

(c) $\frac{dv}{dt} = \frac{v+150}{300}$

$\frac{dv}{ds} \cdot \frac{ds}{dt} = \frac{v+150}{300}$ A1

$v\frac{dv}{ds} = \frac{v+150}{300}$ A1

$\frac{300v}{v+150} dv = ds$ M1

$\int \frac{300v}{v+150} dv = \int ds$ A1

$\int \frac{300(v+150-150)}{v+150} dv = \int ds$ M1

$\int \left(300 - \frac{150}{v+150}\right) dv = \int ds$

$s = \int \left(300 - \frac{150}{v+150}\right) dv$ AG

[5]

(d) $s = \int \left(300 - \frac{150}{v+150}\right) dv$

$s = 300v - 150\ln|v+150| + D$ A1

$0 = 300(0) - 150\ln(0+150) + D$ (M1) for substitution

$D = 150\ln 150$ (A1) for correct value

$\therefore s = 300v - 150\ln(v+150) + 150\ln 150$

$s = 300(5) - 150\ln(5+150) + 150\ln 150$ (M1) for substitution

$s = 1500 - 150(\ln 155 - \ln 150)$

$s = 150\left(10 - \ln\frac{31}{30}\right)$ m A1

[5]

3. (a) (i) Let $\frac{1}{x(x+3)} \equiv \frac{A}{x} + \frac{B}{x+3}$, where A and B are constants.

$$\frac{1}{x(x+3)} \equiv \frac{A(x+3)}{x(x+3)} + \frac{Bx}{x(x+3)}$$ M1

$$\frac{1}{x(x+3)} \equiv \frac{Ax+3A+Bx}{x(x+3)}$$

$$1 \equiv (A+B)x + 3A$$ A1

$$1 = 3A$$

$$A = \frac{1}{3}$$ A1

$$0 = \frac{1}{3} + B$$

$$B = -\frac{1}{3}$$

$$\therefore \frac{1}{x(x+3)} \equiv \frac{1}{3x} - \frac{1}{3(x+3)}$$ A1

(ii) $$\frac{dv}{dt} = -\frac{v(v+3)}{3}$$

$$\frac{1}{v(v+3)} dv = -\frac{1}{3} dt$$ (M1) for valid approach

$$\int \frac{1}{v(v+3)} dv = \int -\frac{1}{3} dt$$ (A1) for correct approach

$$\therefore \frac{1}{3} \int \left(\frac{1}{v} - \frac{1}{v+3} \right) dv = \int -\frac{1}{3} dt$$ (M1) for substitution

$$\int \left(\frac{1}{v} - \frac{1}{v+3} \right) dv = \int -dt$$

$$\ln|v| - \ln|v+3| = -t + C$$ A1

$$\ln\left| \frac{v}{v+3} \right| = -t + C$$

$$\frac{v}{v+3} = e^{-t+C}$$ (A1) for correct approach

$$v = (v+3)e^{-t+C}$$

$$v = ve^{-t+C} + 3e^{-t+C}$$

$$v - ve^{-t+C} = 3e^{-t+C}$$ (M1) for valid approach

$$v(1-e^{-t+C}) = 3e^{-t+C}$$

$$v = \frac{3e^{-t+C}}{1-e^{-t+C}}$$ A1

$$1.5 = \frac{3e^{0+C}}{1-e^{0+C}}$$ (M1) for substitution

$$1.5(1-e^{C}) = 3e^{C}$$

$$1.5 - 1.5e^{C} = 3e^{C}$$

$$1.5 = 4.5e^{C}$$

$$e^{C} = \frac{1}{3}$$ (A1) for correct value

$$\therefore v = \frac{3e^{-t}\left(\frac{1}{3}\right)}{1-e^{-t}\left(\frac{1}{3}\right)}$$

$$v = \frac{3e^{-t}}{3-e^{-t}}$$ A1

[14]

(b) $$\frac{dv}{dt} = -\frac{v(v+3)}{3}$$

$$\frac{dv}{ds}\cdot\frac{ds}{dt} = -\frac{v(v+3)}{3}$$ A1

$$v\frac{dv}{ds} = -\frac{v(v+3)}{3}$$ A1

$$-\frac{3}{v+3}dv = ds$$ M1

$$\int -\frac{3}{v+3}dv = \int ds$$ A1

$$s = -3\ln|v+3| + D$$ A1

$$-2\ln 4.5 = -3\ln(1.5+3) + D$$ M1

$$-2\ln 4.5 = -3\ln 4.5 + D$$

$$D = \ln 4.5$$ A1

$$\therefore s = -3\ln(v+3) + \ln 4.5$$

$$s = \ln\frac{1}{(v+3)^3} + \ln 4.5$$ A1

$$s = \ln\frac{9}{2(v+3)^3}$$ AG

[8]

4. (a) (i) Let $\dfrac{1}{x^2+4x} \equiv \dfrac{A}{x} + \dfrac{B}{x+4}$, where A and B are constants.

$\dfrac{1}{x^2+4x} \equiv \dfrac{A(x+4)}{x(x+4)} + \dfrac{Bx}{x(x+4)}$ M1

$\dfrac{1}{x^2+4x} \equiv \dfrac{Ax+4A+Bx}{x(x+4)}$

$1 \equiv (A+B)x + 4A$ A1

$1 = 4A$

$A = \dfrac{1}{4}$ A1

$0 = \dfrac{1}{4} + B$

$B = -\dfrac{1}{4}$

$\therefore \dfrac{1}{x^2+4x} \equiv \dfrac{1}{4x} - \dfrac{1}{4(x+4)}$ A1

(ii) $\dfrac{\mathrm{d}a}{\mathrm{d}t} = \dfrac{a^2+4a}{4}$

$\dfrac{1}{a^2+4a}\mathrm{d}a = \dfrac{1}{4}\mathrm{d}t$ (M1) for valid approach

$\int \dfrac{1}{a^2+4a}\mathrm{d}a = \int \dfrac{1}{4}\mathrm{d}t$ (A1) for correct approach

$\therefore \dfrac{1}{4}\int\left(\dfrac{1}{a} - \dfrac{1}{a+4}\right)\mathrm{d}a = \int \dfrac{1}{4}\mathrm{d}t$ (M1) for substitution

$\int\left(\dfrac{1}{a} - \dfrac{1}{a+4}\right)\mathrm{d}a = \int \mathrm{d}t$

$\ln|a| - \ln|a+4| = t + C$ A1

$\ln\left|\dfrac{a}{a+4}\right| = t + C$

$\dfrac{a}{a+4} = e^{t+C}$ (A1) for correct approach

$a = (a+4)e^{t+C}$

$a = ae^{t+C} + 4e^{t+C}$

$a - ae^{t+C} = 4e^{t+C}$ (M1) for valid approach

$a(1-e^{t+C}) = 4e^{t+C}$

$a = \dfrac{4e^{t+C}}{1-e^{t+C}}$ A1

$$\frac{4}{e^2-1}=\frac{4e^{0+C}}{1-e^{0+C}}$$ (M1) for substitution

$$\frac{4}{e^2-1}=\frac{4e^{C}}{1-e^{C}}$$

$$\frac{4}{e^2-1}=\frac{4}{e^{-C}-1}$$

$2=-C$

$C=-2$ (A1) for correct value

$\therefore a=\dfrac{4e^{t-2}}{1-e^{t-2}}$ A1

[14]

(b) $a=\dfrac{4e^{t-2}}{1-e^{t-2}}$

$\dfrac{dv}{dt}=\dfrac{4e^{t-2}}{1-e^{t-2}}$

$v=\int\dfrac{4e^{t-2}}{1-e^{t-2}}\,dt$ A1

Let $u=1-e^{t-2}$. M1

$\dfrac{du}{dt}=-e^{t-2}\Rightarrow -1\cdot du=e^{t-2}dt$

$\therefore v=\int-\dfrac{4}{u}\,du$ A1

$v=-4\ln u+D$ A1

$v=-4\ln\left|1-e^{t-2}\right|+D$ A1

$8-4\ln(1-e^{-2})=-4\ln\left|1-e^{0-2}\right|+D$ M1

$8-4\ln(1-e^{-2})+4\ln(1-e^{-2})=D$

$D=8$

$v=-4\ln\left|1-e^{t-2}\right|+8$ A1

$-2\le t-2<0$

$e^{-2}\le e^{t-2}<1$

$0<1-e^{t-2}\le 1-e^{-2}<1$ M1

$\ln(1-e^{t-2})<0$ for $0\le t<2$

$\therefore v=-4\ln\left|1-e^{t-2}\right|+8>0$ for $0\le t<2$ R1

Thus, the particle never stops in $0\le t<2$. AG

[9]

Chapter 16 Solution

Exercise 71

1. (a) The number of different words

$= 5!$ (A1) for correct factorial

$= 120$ A1

[2]

(b) The number of different words

$= 2! \times 4!$ (A2) for correct factorials

$= 48$ A1

[3]

(c) The number of different words

$= 3 \times 4!$ (A2) for correct factorial

$= 72$ A1

[3]

2. (a) The number of different arrangements

$= 2! \times 7!$ (A2) for correct factorials

$= 10080$ A1

[3]

(b) The number of different arrangements

$= 8! - 10080$ (A1) for correct formula

$= 30240$ A1

[2]

(c) The number of different arrangements

$= 2! \times 6! \times 2$ (A2) for correct factorials

$= 2880$ A1

[3]

3. (a) The total number of possible ways

$= \frac{8!}{8}$ (A1) for correct factorial

$= 5040$ A1

[2]

(b) The total number of possible ways

$= \frac{2! \times 6!}{8}$ (A2) for correct factorials

$= 180$ A1

[3]

4. (a) The total number of possible ways

$=\dfrac{10!}{10\times 2}$ (A2) for correct formula

$=181440$ A1

[3]

(b) The number of possible ways

$=181440-\dfrac{2!\times 9!}{9\times 2}$ (A2) for correct formula

$=141120$ A1

[3]

Exercise 72

1. (a) The number of possible teams

$= \binom{10}{4}$ (M1) for valid approach

$= 210$ A1

[2]

(b) The number of teams

$= \binom{5}{1} \times \binom{5}{3}$ M1

$= 50$ A1

[2]

(c) The number of teams

$= \binom{5}{4} \times \binom{5}{0} + \binom{5}{3} \times \binom{5}{1} + \binom{5}{2} \times \binom{5}{2}$ M1

$= 155$ A1

[2]

2. (a) The number of possible teams

$= \binom{16}{6}$ (M1) for valid approach

$= 8008$ A1

[2]

(b) The number of teams

$= \binom{9}{4} \times \binom{7}{2}$ M1

$= 2646$ A1

[2]

(c) The number of teams

$= \binom{9}{1} \times \binom{7}{5} + \binom{9}{3} \times \binom{7}{3} + \binom{9}{5} \times \binom{7}{1}$ M1

$= 4011$ A1

[2]

3. (a) The number of queues

$=5!\times 7!\times 2$ M1A1

$=1209600$ A1

[3]

(b) The number of possible selections

$=\binom{5}{0}\times\binom{7}{6}+\binom{5}{1}\times\binom{7}{5}$ M1

$=112$ A1

[2]

4. (a) $\binom{10}{4}\times\binom{6}{r}\times\binom{6-r}{6-r}=1260$ M1A1

$\binom{6}{r}=6$ (A1) for simplification

$\binom{6}{r}=\binom{6}{1}$ or $\binom{6}{r}=\binom{6}{5}$

$r=1$ or $r=5$ A2

[5]

(b) 1 A1

[1]

Chapter 17 Solution

Exercise 73

1. (a) The required probability

$$= \frac{2}{6} + \left(\frac{4}{6}\right)\left(\frac{4}{6}\right)\left(\frac{2}{6}\right) + \left(\left(\frac{4}{6}\right)\left(\frac{4}{6}\right)\right)^2\left(\frac{2}{6}\right)$$ M1A1

$$= \frac{1}{3} + \frac{4}{27} + \frac{16}{243}$$

$$= \frac{81}{243} + \frac{36}{243} + \frac{16}{243}$$

$$= \frac{133}{243}$$ A1

[3]

(b) The required probability

$$= \frac{2}{6} + \left(\frac{4}{6}\right)\left(\frac{4}{6}\right)\left(\frac{2}{6}\right) + \left(\left(\frac{4}{6}\right)\left(\frac{4}{6}\right)\right)^2\left(\frac{2}{6}\right) + \cdots$$ M1A1

$$= \frac{1}{3}\left(1 + \frac{4}{9} + \left(\frac{4}{9}\right)^2 + \cdots\right)$$

$$= \frac{1}{3}\left(\frac{1}{1 - \frac{4}{9}}\right)$$ A1

$$= \frac{1}{3}\left(\frac{9}{5}\right)$$

$$= \frac{3}{5}$$ A1

[4]

2. (a) The required probability

$$=\left(\left(\frac{4}{5}\right)\left(\frac{4}{5}\right)\right)^2\left(\frac{1}{5}\right)$$ M1A1

$$=\left(\frac{16}{25}\right)\left(\frac{16}{25}\right)\left(\frac{1}{5}\right)$$

$$=\frac{256}{3125}$$ A1

[3]

(b) The required probability

$$=\left(\frac{4}{5}\right)\left(\frac{1}{5}\right)+\left(\frac{4}{5}\right)\left(\frac{4}{5}\right)\left(\frac{4}{5}\right)\left(\frac{1}{5}\right)$$ M1A1

$$+\left(\left(\frac{4}{5}\right)\left(\frac{4}{5}\right)\right)^2\left(\frac{4}{5}\right)\left(\frac{1}{5}\right)+\cdots$$

$$=\frac{4}{25}\left(1+\frac{16}{25}+\left(\frac{16}{25}\right)^2+\cdots\right)$$

$$=\frac{4}{25}\left(\frac{1}{1-\frac{16}{25}}\right)$$ A1

$$=\frac{4}{25}\left(\frac{25}{9}\right)$$

$$=\frac{4}{9}$$ A1

[4]

3. (a) The required probability

$$= \frac{1}{5} + \left(\frac{4}{5}\right)\left(\frac{3}{4}\right)\left(\frac{1}{3}\right) + \left(\frac{4}{5}\right)\left(\frac{3}{4}\right)\left(\frac{2}{3}\right)\left(\frac{1}{2}\right)\left(\frac{1}{1}\right)$$ M1A1

$$= \frac{1}{5} + \frac{1}{5} + \frac{1}{5}$$

$$= \frac{3}{5}$$ A1

[3]

(b) The required probability

$$= \frac{1}{5} + \left(\frac{4}{5}\right)\left(\frac{4}{5}\right)\left(\frac{1}{5}\right) + \left(\left(\frac{4}{5}\right)\left(\frac{4}{5}\right)\right)^2\left(\frac{1}{5}\right) + \cdots$$ M1A1

$$= \frac{1}{5}\left(1 + \frac{16}{25} + \left(\frac{16}{25}\right)^2 + \cdots\right)$$

$$= \frac{1}{5}\left(\frac{1}{1 - \frac{16}{25}}\right)$$ A1

$$= \frac{1}{5}\left(\frac{25}{9}\right)$$

$$= \frac{5}{9}$$ A1

[4]

4. (a) The required probability

$$=\frac{1}{4}+\left(\frac{3}{4}\right)\left(\frac{3}{4}\right)\left(\frac{1}{4}\right)+\left(\left(\frac{3}{4}\right)\left(\frac{3}{4}\right)\right)^2\left(\frac{1}{4}\right)+\cdots$$ M1A1

$$=\frac{1}{4}\left(1+\frac{9}{16}+\left(\frac{9}{16}\right)^2+\cdots\right)$$

$$=\frac{1}{4}\left(\frac{1}{1-\frac{9}{16}}\right)$$ A1

$$=\frac{1}{4}\left(\frac{16}{7}\right)$$

$$=\frac{4}{7}$$ A1

[4]

(b) $$\left(\left(\frac{3}{4}\right)\left(\frac{3}{4}\right)\right)^{n-1}\left(\frac{3}{4}\right)\left(\frac{1}{4}\right)=\frac{243}{4096}$$ M1A1

$$\left(\frac{9}{16}\right)^{n-1}\left(\frac{3}{16}\right)=\frac{243}{4096}$$

$$\left(\frac{9}{16}\right)^{n-1}=\frac{81}{256}$$

$$\left(\frac{9}{16}\right)^{n-1}=\left(\frac{9}{16}\right)^2$$ (A1) for correct equation

$\therefore n-1=2$

$n=3$ A1

[4]

Exercise 74

1. (a) The required probability

$$= \left(\frac{50}{50+36}\right)(11\%) + \left(\frac{36}{50+36}\right)(17\%)$$ M1A1

$$= \frac{581}{4300}$$ A1

[3]

(b) The required probability

$$= \frac{\left(\frac{50}{50+36}\right)(11\%)}{\frac{581}{4300}}$$ M1A1

$$= \frac{275}{581}$$ A1

[3]

2. (a) Let p be the required probability.

$$p\left(\frac{11}{40}\right) + (1-p)\left(\frac{3}{40}\right) = 0.121$$ M1A1

$$\frac{11}{40}p + \frac{3}{40} - \frac{3}{40}p = 0.121$$

$$\frac{1}{5}p = 0.046$$

$$p = 0.23$$ A1

[3]

(b) The required probability

$$= \frac{(0.23)\left(1-\frac{11}{40}\right)}{1-0.121}$$ M1A1

$$= \frac{667}{3516}$$ A1

[3]

3. (a) $(1.5\%)\left(\frac{100p}{100}\right)+(1-1.5\%)\left(\frac{p}{100}\right)=0.024353$ M1A1

$0.015p+0.00985p=0.024353$

$0.02485p=0.024353$

$p=0.98$ A1

[3]

(b) The required probability

$=\frac{(1-1.5\%)\left(\frac{0.98}{100}\right)}{0.024353}$ M1A1

$=\frac{197}{497}$ A1

[3]

4. The required probability

$=\frac{\left(\frac{2}{7}\right)(0.465)}{\left(\frac{5}{7}\right)(0.675)+\left(\frac{2}{7}\right)(0.465)}$ M1A2

$=\frac{62}{287}$ A1

[4]

Exercise 75

1. (a) The required probability

$$= \left(\frac{35}{145}\right)\left(\frac{20}{35}\right) + \left(\frac{60}{145}\right)\left(\frac{18}{60}\right) + \left(\frac{145-35-60}{145}\right)\left(\frac{3}{5}\right) \quad \text{M1A1}$$

$$= \frac{68}{145} \quad \text{A1}$$

[3]

(b) The required probability

$$= \frac{\left(\frac{145-35-60}{145}\right)\left(\frac{3}{5}\right)}{\frac{68}{145}} \quad \text{M1A1}$$

$$= \frac{15}{34} \quad \text{A1}$$

[3]

2. (a) $(2p)(45\%) + (p)(15\%) + (1-2p-p)(8\%) = 0.2015$ M1A1

$0.9p + 0.15p + 0.08 - 0.24p = 0.2015$

$0.81p = 0.1215$

$p = 0.15$ A1

[3]

(b) The required probability

$$= \frac{(1-0.3-0.15)(1-8\%)}{1-0.2015} \quad \text{M1A1}$$

$$= \frac{1012}{1597} \quad \text{A1}$$

[3]

3. (a) $\left(\frac{50}{160}\right)(2q\%)+\left(\frac{160-50}{160}\right)(q\%)=0.105$ M1A1

$\left(\frac{50}{160}\right)\left(\frac{2q}{100}\right)+\left(\frac{110}{160}\right)\left(\frac{q}{100}\right)=0.105$

$\frac{21}{1600}q=0.105$

$q=8$ A1

[3]

(b) The required probability

$$=\frac{\left(\frac{110}{160}\right)(1-12\%-8\%)}{\left(\frac{50}{160}\right)(1-8\%-16\%)+\left(\frac{110}{160}\right)(1-12\%-8\%)}$$ M1A1

$=\frac{44}{63}$ A1

[3]

4. The required probability

$$=\frac{\left(\frac{11}{7+11+2}\right)(1\%)+\left(\frac{2}{7+11+2}\right)(3\%)}{\left(\frac{7}{7+11+2}\right)(2\%)+\left(\frac{11}{7+11+2}\right)(1\%)+\left(\frac{2}{7+11+2}\right)(3\%)}$$ M1A2

$=\frac{17}{31}$ A1

[4]

Exercise 76

1. (a) The required probability

$$= \frac{\binom{7}{6}\times\binom{10}{2}}{\binom{17}{8}}$$ M1A1

$$= \frac{63}{4862}$$ A1

[3]

(b) The required probability

$$= \frac{\binom{7}{6}\times\binom{10}{2}+\binom{7}{7}\times\binom{10}{1}}{\binom{17}{8}}$$ M1A1

$$= \frac{5}{374}$$ A1

[3]

2. (a) The required probability

$$= \frac{\binom{15}{1}\times\binom{4}{4}}{\binom{19}{5}}$$ M1A1

$$= \frac{5}{3876}$$ A1

[3]

(b) The required probability

$$= \frac{\binom{15}{1}\times\binom{4}{4}+\binom{15}{3}\times\binom{4}{2}+\binom{15}{5}\times\binom{4}{0}}{\binom{19}{5}}$$ M1A1

$$= \frac{479}{969}$$ A1

[3]

3. (a) The required probability

$$= \frac{8! \times 6!}{13!}$$ (A2) for correct factorials

$$= \frac{2}{429}$$ A1

[3]

(b) The required probability

$$= \frac{{}^{9}P_{5} \times 8!}{13!}$$ (A2) for correct factorials

$$= \frac{14}{143}$$ A1

[3]

4. The required probability

$$= \frac{\frac{5! \times 5!}{10}}{\frac{10!}{10}}$$ (A3) for correct factorials

$$= \frac{1}{252}$$ A1

[4]

Chapter 18 Solution

Exercise 77

1. (a) $P(X=1)+P(X=2)+P(X=5)+P(X=10)=1$ (A1) for correct formula

$(r+s)+(r-s)+0.15+0.05=1$

$2r=0.8$

$r=0.4$ A1

$E(X)=(1)(r+s)+(2)(r-s)+(5)(0.15)+(10)(0.05)$

$\therefore 2.35=0.4+s+2(0.4-s)+0.75+0.5$ (A1) for substitution

$2.35=2.45-s$

$s=0.1$ A1

[4]

(b) (i) -2.35 A1

(ii) 4.9275 A1

[2]

2. (a) $P(X=10)+P(X=20)+P(X=t)+P(X=2t)=1$ (A1) for correct formula

$0.2+0.5+r+r^2+0.06=1$

$r^2+r-0.24=0$

$25r^2+25r-6=0$

$(5r+6)(5r-1)=0$

$r=-1.2$ (*Rejected*) or $r=0.2$ A1

$E(X)=(10)(0.2)+(20)(0.5)+tr+(2t)(r^2+0.06)$

$\therefore 24=2+10+0.2t+0.2t$ (A1) for substitution

$12=0.4t$

$t=30$ A1

[4]

(b) $4\sqrt{184}=q\sqrt{46}$ A1

$q=\dfrac{4\sqrt{184}}{\sqrt{46}}$

$q=4(2)$

$q=8$ A1

[2]

3. (a) $P(X=-2)+P(X=0)+P(X=2)+P(X=4)=1$ (A1) for correct formula

$r^2+0.21+r+0.25+0.15=1$

$r^2+r-0.39=0$

$100r^2+100r-39=0$

$(10r+13)(10r-3)=0$

$r=-1.3$ (*Rejected*) or $r=0.3$ A1

[2]

(b) $E(X)=(-2)(r^2+0.21)+0+(2)(0.25)+(4)(0.15)$ (A1) for correct formula

$E(X)=-0.6+0+0.5+0.6$

$E(X)=0.5$ A1

[2]

(c) $\text{Var}(Y)=a^2\text{Var}(X)$ A1

$17.4=4.35a^2$

$a^2=4$

$a=-2$ (*Rejected*) or $a=2$ A1

[2]

(d) 4.5 A1

[1]

4. (a) $P(X=5)+P(X=10)+P(X=15)+P(X=20)=1$ (A1) for correct formula

$0.4+0.4-r+0.19+r^2+r=1$

$r^2=0.01$

$r=-0.1$ (*Rejected*) or $r=0.1$ A1

[2]

(b) $E(X)=(5)(0.4)+(10)(0.4-r)$
$+(15)(0.19+r^2)+20r$ (A1) for correct formula

$E(X)=2+3+3+2$

$E(X)=10$ A1

[2]

(c) $a=\frac{\sqrt{25}}{1}$ A1

$a=5$ A1

[2]

(d) $E(X)=5E(Y)+20$

$10=5E(Y)+20$ (A1) for substitution

$5E(Y)=-10$

$E(Y)=-2$ A1

[2]

Exercise 78

1.	(a)	$E(X) = (18)(0.343) + (24)(0.189)$ $+ (33)(0.0265) + (39)(0.4415)$	(A1) for correct formula
		$E(X) = 28.803$	A1
			[2]
	(b)	$E(X^2) = (18^2)(0.343) + (24^2)(0.189)$ $+ (33^2)(0.0265) + (39^2)(0.4415)$	(A1) for correct formula
		$E(X^2) = 920.376$	A1
			[2]
	(c)	The standard deviation of X	
		$= \sqrt{E(X^2) - E(X)^2}$	(A1) for correct formula
		$= \sqrt{920.376 - 28.803^2}$	
		$= 9.526971764$	
		$= 9.53$	A1
			[2]
2.	(a)	$E(X) = (4)(0.45) + (9)(0.2) + (16)(0.2) + (25)(0.15)$	(A1) for correct formula
		$E(X) = 10.55$	A1
			[2]
	(b)	$E(Y) = (-\sqrt{4})(0.45) + (-\sqrt{9})(0.2)$ $+ (-\sqrt{16})(0.2) + (-\sqrt{25})(0.15)$	(A1) for correct formula
		$E(Y) = -3.05$	A1
			[2]
	(c)	The standard deviation of Y	
		$= \sqrt{E(Y^2) - E(Y)^2}$	(A1) for correct formula
		$= \sqrt{E(X) - E(Y)^2}$	
		$= \sqrt{10.55 - (-3.05)^2}$	
		$= 1.116915395$	
		$= 1.12$	A1
			[2]

3. (a)

$$P(X=1)+P(X=2)+P(X=3)$$
$$+P(X=4)+P(X=5)+P(X=6)=1$$

(A1) for correct formula

$$\frac{1}{42k}+\frac{2}{42k}+\frac{3}{42k}+\frac{4}{42k}+\frac{5}{42k}+\frac{6}{42k}=1$$

$$\frac{21}{42k}=1$$

$$k=\frac{1}{2}$$

A1

[2]

(b)

$$E(X)=(1)\left(\frac{1}{21}\right)+(2)\left(\frac{2}{21}\right)+(3)\left(\frac{3}{21}\right)$$
$$+(4)\left(\frac{4}{21}\right)+(5)\left(\frac{5}{21}\right)+(6)\left(\frac{6}{21}\right)$$

(A1) for correct formula

$$E(X)=\frac{13}{3}$$

A1

[2]

(c)

$$\mathrm{Var}(X)=E(X^2)-E(X)^2$$

(A1) for correct formula

$$\mathrm{Var}(X)=(1^2)\left(\frac{1}{21}\right)+(2^2)\left(\frac{2}{21}\right)+(3^2)\left(\frac{3}{21}\right)$$
$$+(4^2)\left(\frac{4}{21}\right)+(5^2)\left(\frac{5}{21}\right)+(6^2)\left(\frac{6}{21}\right)-\left(\frac{13}{3}\right)^2$$

(A1) for substitution

$$\mathrm{Var}(X)=\frac{20}{9}$$

A1

[3]

4. (a)

$$P(X=3)+P(X=4)+P(X=5)$$
$$+P(X=6)+P(X=7)=1$$ (A1) for correct formula

$$\left(\frac{3}{16}-\frac{9}{5k^2}\right)+\left(\frac{4}{16}-\frac{9}{5k^2}\right)+\left(\frac{5}{16}-\frac{9}{5k^2}\right)$$
$$+\left(\frac{6}{16}-\frac{9}{5k^2}\right)+\left(\frac{7}{16}-\frac{9}{5k^2}\right)=1$$

$$\frac{25}{16}-\frac{9}{k^2}=1$$ (M1) for valid approach

$$-\frac{9}{k^2}=-\frac{9}{16}$$

$$k^2=16$$

$k=-4$ (*Rejected*) or $k=4$ A1

[3]

(b)

$$\text{Var}(X)=E(X^2)-E(X)^2$$ (A1) for correct formula

$$\text{Var}(X)=(3^2)\left(\frac{3}{40}\right)+(4^2)\left(\frac{11}{80}\right)$$
$$+(5^2)\left(\frac{1}{5}\right)+(6^2)\left(\frac{21}{80}\right)+(7^2)\left(\frac{13}{40}\right)$$ (A2) for substitution
$$-\left((3)\left(\frac{3}{40}\right)+(4)\left(\frac{11}{80}\right)+(5)\left(\frac{1}{5}\right)+(6)\left(\frac{21}{80}\right)+(7)\left(\frac{13}{40}\right)\right)^2$$

$$\text{Var}(X)=\frac{133}{4}-\left(\frac{45}{8}\right)^2$$

$$\text{Var}(X)=\frac{103}{64}$$ A1

[4]

Chapter 19 Solution

Exercise 79

1. (a) $E(X)=\int_1^3 x\cdot\left(-\frac{1}{2}x+\frac{3}{2}\right)dx$ (A1) for substitution

$E(X)=\int_1^3\left(-\frac{1}{2}x^2+\frac{3}{2}x\right)dx$

$E(X)=\left[-\frac{1}{6}x^3+\frac{3}{4}x^2\right]_1^3$

$E(X)=\left(-\frac{27}{6}+\frac{27}{4}\right)-\left(-\frac{1}{6}+\frac{3}{4}\right)$

$E(X)=\frac{5}{3}$ A1

[2]

(b) $\int_1^a \left(-\frac{1}{2}x+\frac{3}{2}\right)\mathrm{d}x=\frac{1}{4}$ A1

$\left[-\frac{1}{4}x^2+\frac{3}{2}x\right]_1^a=\frac{1}{4}$ A1

$\left(-\frac{1}{4}a^2+\frac{3}{2}a\right)-\left(-\frac{1}{4}+\frac{3}{2}\right)=\frac{1}{4}$

$-\frac{1}{4}a^2+\frac{3}{2}a-\frac{5}{4}-\frac{1}{4}=0$

$a^2-6a+6=0$

$a=\frac{-(-6)\pm\sqrt{(-6)^2-4(1)(6)}}{2(1)}$

$a=\frac{6+\sqrt{12}}{2}$ (*Rejected*) or $a=\frac{6-\sqrt{12}}{2}$ (A1) for correct value

$\int_1^b \left(-\frac{1}{2}x+\frac{3}{2}\right)\mathrm{d}x=\frac{3}{4}$

$-\frac{1}{4}b^2+\frac{3}{2}b-\frac{5}{4}-\frac{3}{4}=0$

$b^2-6b+8=0$

$(b-2)(b-4)=0$

$b=2$ or $b=4$ (*Rejected*) (A1) for correct value

The interquartile range

$=2-\frac{6-\sqrt{12}}{2}$

$=2-(3-\sqrt{3})$

$=-1+\sqrt{3}$ A1

[5]

2. (a) $\mathrm{E}(X)=\int_{-1}^{2} x\cdot\frac{1}{3}\mathrm{d}x$ (A1) for substitution

$\mathrm{E}(X)=\left[\frac{1}{6}x^2\right]_{-1}^{2}$

$\mathrm{E}(X)=\frac{4}{6}-\frac{1}{6}$

$\mathrm{E}(X)=\frac{1}{2}$ A1

[2]

(b) $\mathrm{E}(X^2)=\int_{-1}^{2} x^2\cdot\frac{1}{3}\mathrm{d}x$ (A1) for substitution

$\mathrm{E}(X^2)=\left[\frac{1}{9}x^3\right]_{-1}^{2}$

$\mathrm{E}(X^2)=\frac{8}{9}-\left(-\frac{1}{9}\right)$

$\mathrm{E}(X^2)=1$ A1

[2]

(c) Standard deviation

$=\mathrm{E}(X^2)-(\mathrm{E}(X))^2$ (A1) for substitution

$=1-\left(\frac{1}{2}\right)^2$

$=\frac{3}{4}$ A1

[2]

3. (a) 2 A1

[1]

(b) $\int_0^a \frac{1}{e^2-1}e^x \mathrm{d}x = \frac{1}{2}$ A1

$\left[\frac{1}{e^2-1}e^x\right]_0^a = \frac{1}{2}$ A1

$\frac{1}{e^2-1}e^a - \frac{1}{e^2-1}e^0 = \frac{1}{2}$

$\frac{e^a-1}{e^2-1} = \frac{1}{2}$ (M1) for valid approach

$e^a - 1 = \frac{1}{2}e^2 - \frac{1}{2}$

$e^a = \frac{e^2+1}{2}$

$a = \ln\left(\frac{e^2+1}{2}\right)$

Thus, the median is $\ln\left(\frac{e^2+1}{2}\right)$. A1

[4]

4. (a) $\sqrt{3}$ A1

[1]

(b) $\int_{-1}^{a}\left|\frac{x}{2}\right|\mathrm{d}x=\frac{1}{2}$ A1

$\int_{-1}^{0}-\frac{x}{2}\mathrm{d}x+\int_{0}^{a}\frac{x}{2}\mathrm{d}x=\frac{1}{2}$

$\left[-\frac{1}{4}x^2\right]_{-1}^{0}+\left[\frac{1}{4}x^2\right]_{0}^{a}=\frac{1}{2}$ A1

$\left(0-\left(-\frac{1}{4}\right)\right)+\left(\frac{1}{4}a^2-0\right)=\frac{1}{2}$

$\frac{1}{4}a^2=\frac{1}{4}$ (M1) for valid approach

$a^2=1$

$a=-1$ (*Rejected*) or $a=1$

Thus, the median is 1. A1

[4]

(c) $\mathrm{E}(X)=\int_{-1}^{\sqrt{3}}x\cdot\left|\frac{x}{2}\right|\mathrm{d}x$ (A1) for substitution

$\mathrm{E}(X)=\int_{-1}^{0}-\frac{1}{2}x^2\mathrm{d}x+\int_{0}^{\sqrt{3}}\frac{1}{2}x^2\mathrm{d}x$

$\mathrm{E}(X)=\left[-\frac{1}{6}x^3\right]_{-1}^{0}+\left[\frac{1}{6}x^3\right]_{0}^{\sqrt{3}}$ A1

$\mathrm{E}(X)=\left(0-\frac{1}{6}\right)+\left(\frac{3\sqrt{3}}{6}-0\right)$

$\mathrm{E}(X)=\frac{3\sqrt{3}-1}{6}$ A1

[3]

Exercise 80

1. (a) $\int_{1}^{\frac{3}{2}} k(x-1)\mathrm{d}x + \int_{\frac{3}{2}}^{2} k(x-2)^2\mathrm{d}x = 1$ (M1) for valid approach

$\left[\frac{k}{2}(x-1)^2\right]_1^{\frac{3}{2}} + \left[\frac{k}{3}(x-2)^3\right]_{\frac{3}{2}}^{2} = 1$ A1

$\left(\frac{k}{8}-0\right)+\left(0-\left(-\frac{k}{24}\right)\right)=1$

$\frac{k}{6}=1$

$k=6$ A1

[3]

(b) $\mathrm{E}(X) = \int_{1}^{\frac{3}{2}} 6x(x-1)\mathrm{d}x + \int_{\frac{3}{2}}^{2} 6x(x-2)^2\mathrm{d}x$ (A1) for substitution

$\mathrm{E}(X) = 1+\frac{13}{32}$ (A1) for correct values

$\mathrm{E}(X) = \frac{45}{32}$ A1

[3]

(c) $\int_{1}^{a} 6(x-1)\mathrm{d}x = \frac{1}{2}$ A1

$\left[3(x-1)^2\right]_1^a = \frac{1}{2}$ A1

$3(a-1)^2 - 0 = \frac{1}{2}$

$(a-1)^2 = \frac{1}{6}$

$a-1=\sqrt{\frac{1}{6}}$

$a=1.40824829$

Thus, the median is 1.41. A1

[3]

(d) $\mathrm{P}\left(X>\frac{5}{4}\right)=\int_{\frac{5}{4}}^{\frac{3}{2}} 6(x-1)\mathrm{d}x+\int_{\frac{3}{2}}^{2} 6(x-2)^2\mathrm{d}x$ (M1) for valid approach

$\mathrm{P}\left(X>\frac{5}{4}\right)=\frac{9}{16}+\frac{1}{4}$ (A1) for correct values

$\mathrm{P}\left(X>\frac{5}{4}\right)=\frac{13}{16}$ A1

[3]

(e) $\mathrm{P}\left(X>\frac{7}{4}\middle| X>\frac{5}{4}\right)=\frac{\mathrm{P}\left(X>\frac{7}{4}\cap X>\frac{5}{4}\right)}{\mathrm{P}\left(X>\frac{5}{4}\right)}$ (M1) for valid approach

$\mathrm{P}\left(X>\frac{7}{4}\middle| X>\frac{5}{4}\right)=\frac{\mathrm{P}\left(X>\frac{7}{4}\right)}{\mathrm{P}\left(X>\frac{5}{4}\right)}$

$\mathrm{P}\left(X>\frac{7}{4}\middle| X>\frac{5}{4}\right)=\frac{\int_{\frac{7}{4}}^{2} 6(x-2)^2\mathrm{d}x}{\frac{13}{16}}$ A1

$\mathrm{P}\left(X>\frac{7}{4}\middle| X>\frac{5}{4}\right)=\frac{16}{13}\left(\frac{1}{32}\right)$

$\mathrm{P}\left(X>\frac{7}{4}\middle| X>\frac{5}{4}\right)=\frac{1}{26}$ A1

[3]

2. (a) $\mathrm{E}(X)=\int_{-\frac{3\pi}{2}}^{-\frac{\pi}{2}}-\frac{1}{\pi^2}x\left(x+\frac{\pi}{2}\right)\mathrm{d}x+\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\frac{1}{4}x\cos x\mathrm{d}x$ (A1) for substitution

$\mathrm{E}(X)=-1.832595715+0$ (A1) for correct values

$\mathrm{E}(X)=-1.832595715$

$\mathrm{E}(X)=-1.83$ A1

[3]

(b) The variance of X

$=\mathrm{E}(X^2)-(\mathrm{E}(X))^2$ (M1) for valid approach

$=\int_{-\frac{3\pi}{2}}^{-\frac{\pi}{2}}-\frac{1}{\pi^2}x^2\left(x+\frac{\pi}{2}\right)\mathrm{d}x+\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\frac{1}{4}x^2\cos x\mathrm{d}x$ A1

$-(-1.832595715)^2$

$=6.990969784+0.2337005501-(-1.832595715)^2$

$=3.866263279$

$=3.87$ A1

[3]

(c) $\mathrm{P}(X<0)=\int_{-\frac{3\pi}{2}}^{-\frac{\pi}{2}}-\frac{1}{\pi^2}\left(x+\frac{\pi}{2}\right)\mathrm{d}x+\int_{-\frac{\pi}{2}}^{0}\frac{1}{4}\cos x\mathrm{d}x$ (M1) for valid approach

$\mathrm{P}(X<0)=\frac{1}{2}+\frac{1}{4}$ (A1) for correct values

$\mathrm{P}(X<0)=\frac{3}{4}$ A1

[3]

(d) $P(X > -\pi \mid X < 0) = \dfrac{P(X > -\pi \cap X < 0)}{P(X < 0)}$ (M1) for valid approach

$$P(X > -\pi \mid X < 0) = \frac{P(-\pi < X < 0)}{P(X < 0)}$$

$$P(X > -\pi \mid X < 0)$$

$$= \frac{\int_{-\pi}^{-\frac{\pi}{2}} -\frac{1}{\pi^2}\left(x + \frac{\pi}{2}\right)dx + \int_{-\frac{\pi}{2}}^{0} \frac{1}{4}\cos x dx}{\frac{3}{4}}$$ A1

$$P(X > -\pi \mid X < 0) = \frac{4}{3}\left(\frac{1}{8} + \frac{1}{4}\right)$$

$P(X > -\pi \mid X < 0) = \dfrac{1}{2}$ A1

[3]

(e) $P(X < r) = 0.6$

$P\left(X < -\dfrac{\pi}{2}\right) + P\left(-\dfrac{\pi}{2} < X < r\right) = 0.6$ (M1) for valid approach

$$\frac{1}{2} + \int_{-\frac{\pi}{2}}^{r} \frac{1}{4}\cos x dx = 0.6$$

$$\left[\frac{1}{4}\sin x\right]_{-\frac{\pi}{2}}^{r} = 0.1$$

$\dfrac{1}{4}\sin r - \left(-\dfrac{1}{4}\right) = 0.1$ A1

$\sin r = -0.6$

$r = -0.6435011088$

$r = -0.644$ A1

[3]

3. (a) $\int_0^{\pi} k\sin\frac{1}{2}x\mathrm{d}x+\int_{\pi}^{\pi+1}k\mathrm{d}x=1$ M1

$\left[-2k\cos\frac{1}{2}x\right]_0^{\pi}+\left[kx\right]_{\pi}^{\pi+1}=1$ A1

$(0-(-2k))+(k(\pi+1)-k\pi)=1$ A1

$3k=1$

$k=\frac{1}{3}$ AG

[3]

(b) $\mathrm{E}(X)=\int_0^{\pi}\frac{1}{3}x\sin\frac{1}{2}x\mathrm{d}x+\int_{\pi}^{\pi+1}\frac{1}{3}x\mathrm{d}x$ (A1) for substitution

$\mathrm{E}(X)=\frac{4}{3}+1.213864218$ (A1) for correct values

$\mathrm{E}(X)=2.547197551$

$\mathrm{E}(X)=2.55$ A1

[3]

(c) $\int_0^{\pi}\frac{1}{3}\sin\frac{1}{2}x\mathrm{d}x+\int_{\pi}^{a}\frac{1}{3}\mathrm{d}x=\frac{3}{4}$ A1

$\left[-\frac{2}{3}\cos\frac{1}{2}x\right]_0^{\pi}+\left[\frac{1}{3}x\right]_{\pi}^{a}=\frac{3}{4}$ A1

$\left(0-\left(-\frac{2}{3}\right)\right)+\left(\frac{a}{3}-\frac{\pi}{3}\right)=\frac{3}{4}$

$\frac{a}{3}-\frac{\pi}{3}=\frac{1}{12}$

$a-\pi=\frac{1}{4}$

$a=\frac{1}{4}+\pi$

Thus, the upper quartile is $\frac{1}{4}+\pi$. A1

[3]

(d) $P\left(X<\frac{\pi}{2}\right)=\int_0^{\frac{\pi}{2}}\frac{1}{3}\sin\frac{1}{2}x\,dx$ M1

$$P\left(X<\frac{\pi}{2}\right)=\left[-\frac{2}{3}\cos\frac{1}{2}x\right]_0^{\frac{\pi}{2}}$$

$P\left(X<\frac{\pi}{2}\right)=-\frac{2}{3}\left(\frac{\sqrt{2}}{2}\right)-\left(-\frac{2}{3}\right)$ A1

$P\left(X<\frac{\pi}{2}\right)=\frac{2-\sqrt{2}}{3}$ AG

[2]

(e) $P\left(\frac{\pi}{2}<X<\pi \middle| X>\frac{\pi}{2}\right)=\frac{P\left(\frac{\pi}{2}<X<\pi\cap X>\frac{\pi}{2}\right)}{P\left(X>\frac{\pi}{2}\right)}$ (M1) for valid approach

$$P\left(\frac{\pi}{2}<X<\pi \middle| X>\frac{\pi}{2}\right)=\frac{P\left(\frac{\pi}{2}<X<\pi\right)}{P\left(X>\frac{\pi}{2}\right)}$$

$P\left(\frac{\pi}{2}<X<\pi \middle| X>\frac{\pi}{2}\right)=\frac{\int_{\frac{\pi}{2}}^{\pi}\frac{1}{3}\sin\frac{1}{2}x\,dx}{1-\frac{2-\sqrt{2}}{3}}$ A1

$$P\left(\frac{\pi}{2}<X<\pi \middle| X>\frac{\pi}{2}\right)=\frac{\left[-\frac{2}{3}\cos\frac{1}{2}x\right]_{\frac{\pi}{2}}^{\pi}}{\frac{1+\sqrt{2}}{3}}$$

$$P\left(\frac{\pi}{2}<X<\pi \middle| X>\frac{\pi}{2}\right)=\frac{0-\left(-\frac{2}{3}\left(\frac{\sqrt{2}}{2}\right)\right)}{\frac{1+\sqrt{2}}{3}}$$

$$P\left(\frac{\pi}{2}<X<\pi \middle| X>\frac{\pi}{2}\right)=\frac{\frac{\sqrt{2}}{3}}{\frac{1+\sqrt{2}}{3}}$$

$P\left(\frac{\pi}{2}<X<\pi \middle| X>\frac{\pi}{2}\right)=\frac{\sqrt{2}}{1+\sqrt{2}}$ A1

[3]

4. (a)

$$E(X)=\int_{-3}^{1}x\cdot\left(\frac{1}{20}x+\frac{3}{20}\right)dx+\int_{1}^{4}x\cdot\left(-\frac{1}{15}x^2+\frac{4}{15}x\right)dx$$ (A1) for substitution

$$E(X)=\int_{-3}^{1}\left(\frac{1}{20}x^2+\frac{3}{20}x\right)dx+\int_{1}^{4}\left(-\frac{1}{15}x^3+\frac{4}{15}x^2\right)dx$$

$$E(X)=-\frac{2}{15}+\frac{27}{20}$$ (A1) for correct values

$$E(X)=\frac{73}{60}$$ A1

[3]

(b) By considering the graphs of $y=\frac{1}{20}x+\frac{3}{20}$, $-3\le x\le 1$ and $y=-\frac{1}{15}x^2+\frac{4}{15}x$, $1<x\le 4$, the maximum point is $\left(2,\frac{4}{15}\right)$. (M1) for valid approach

Thus, the mode of X is 2. A1

[2]

(c) The standard deviation of X

$=\sqrt{E(X^2)-(E(X))^2}$ (M1) for valid approach

$$=\sqrt{\int_{-3}^{1}x^2\cdot\left(\frac{1}{20}x+\frac{3}{20}\right)dx+\int_{1}^{4}x^2\cdot\left(-\frac{1}{15}x^2+\frac{4}{15}x\right)dx-\left(\frac{73}{60}\right)^2}$$ A1

$=\sqrt{2.279722222}$

$=1.509874903$

$=1.51$ A1

[3]

(d) $\int_{-3}^{a}\left(\frac{1}{20}x+\frac{3}{20}\right)\mathrm{d}x=0.35$ A1

$\left[\frac{1}{40}x^2+\frac{3}{20}x\right]_{-3}^{a}=0.35$

$\left(\frac{1}{40}a^2+\frac{3}{20}a\right)-\left(-\frac{9}{40}\right)=0.35$

$a^2+6a+9=14$

$a^2+6a-5=0$ A1

By considering the graph of $y=a^2+6a-5$,

$a=-6.741657387$ (*Rejected*) or $a=0.741657386$.

Thus, the 35th percentile is 0.742. A1

[3]

(e) $\mathrm{P}(|X|>2)=\mathrm{P}(X>2 \text{ or } X<-2)$ (M1) for valid approach

$\mathrm{P}(|X|>2)=\mathrm{P}(X<-2)+\mathrm{P}(X>2)$

$\mathrm{P}(|X|>2)=\int_{-3}^{-2}\left(\frac{1}{20}x+\frac{3}{20}\right)\mathrm{d}x$

$+\int_{2}^{4}\left(-\frac{1}{15}x^2+\frac{4}{15}x\right)\mathrm{d}x$

$\mathrm{P}(|X|>2)=\frac{1}{40}+\frac{16}{45}$ (A1) for correct values

$\mathrm{P}(|X|>2)=\frac{137}{360}$ A1

[3]

(f) $\mathrm{P}(X>2\,|\,|X|>2)=\frac{\mathrm{P}(X>2\cap|X|>2)}{\mathrm{P}(|X|>2)}$ (M1) for valid approach

$\mathrm{P}(X>2\,|\,|X|>2)=\frac{\mathrm{P}(X>2)}{\mathrm{P}(|X|>2)}$

$\mathrm{P}(X>2\,|\,|X|>2)=\frac{\int_{2}^{4}\left(-\frac{1}{15}x^2+\frac{4}{15}x\right)\mathrm{d}x}{\frac{137}{360}}$ A1

$\mathrm{P}(X>2\,|\,|X|>2)=\frac{360}{137}\left(\frac{16}{45}\right)$

$\mathrm{P}(X>2\,|\,|X|>2)=\frac{128}{137}$ A1

[3]

Chapter 20 Solution

Quick Practice

Part I Solution

(a) (1) The area of the rectangle ABCD

$$= (\text{AD})(\text{CD})$$
$$= (e^{-1} - e^{-2})(1)$$
$$= \frac{e-1}{e^2}$$

(2) The area of the triangle ACD

$$= \frac{e-1}{e^2} \div 2$$
$$= \frac{e-1}{2e^2}$$

(3) $0 < A_1 < \frac{e-1}{2e^2}$

(b) (1) The area of the triangle CFG

$$= \frac{(\text{CG})(\text{FG})}{2}$$
$$= \frac{(e^{-2} - e^{-3})(1)}{2}$$
$$= \frac{e^{-2} - e^{-3}}{2}$$
$$= \frac{e-1}{2e^3}$$

(2) $0 < A_2 < \frac{e-1}{2e^3}$

(c) $$0 < A_3 < \frac{(e^{-3} - e^{-4})(1)}{2}$$

$$0 < A_3 < \frac{e^{-3} - e^{-4}}{2}$$

$$0 < A_3 < \frac{e-1}{2e^4}$$

(d) $$0 < A_n < \frac{(e^{-n} - e^{-(n+1)})(1)}{2}$$

$$0 < A_n < \frac{e^{-n} - e^{-n-1}}{2}$$

$$0 < A_n < \frac{e-1}{2e^{n+1}}$$

(e) (1) $$A_n = \int_n^{n+1} e^{-x}\mathrm{d}x - (1)(e^{-(n+1)})$$

$$A_n = \left[-e^{-x}\right]_n^{n+1} - e^{-(n+1)}$$

$$A_n = -e^{-(n+1)} - (-e^{-n}) - e^{-(n+1)}$$

$$A_n = -2e^{-(n+1)} + e^{-n}$$

$$A_n = \frac{e-2}{e^{n+1}}$$

(2) $$A_{n+2} = \frac{e-2}{e^{(n+2)+1}}$$

$$A_{n+2} = \frac{e-2}{e^{n+3}}$$

(3) $$A_n - A_{n+2} = \frac{e-2}{e^{n+1}} - \frac{e-2}{e^{n+3}}$$

$$A_n - A_{n+2} = \frac{e^2(e-2) - (e-2)}{e^{n+3}}$$

$$A_n - A_{n+2} = \frac{(e^2-1)(e-2)}{e^{n+3}}$$

$$A_n - A_{n+2} = \frac{(e+1)(e-1)(e-2)}{e^{n+3}}$$

(f) $e^{23}A_\beta - e + 2 = 0$

$e^{23}A_\beta = e - 2$

$A_\beta = \dfrac{e-2}{e^{23}}$

$\dfrac{e-2}{e^{\beta+1}} = \dfrac{e-2}{e^{23}}$

$\therefore \beta + 1 = 23$

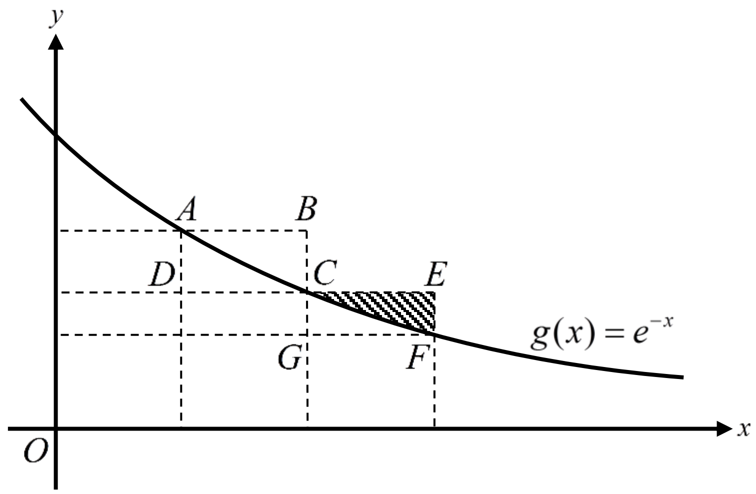
$\beta = 22$

Part II Solution

(g) (1) $\dfrac{e-1}{2e^2} < B_1 < 2\left(\dfrac{e-1}{2e^2}\right)$

$\dfrac{e-1}{2e^2} < B_1 < \dfrac{e-1}{e^2}$

(2)



(3) $\dfrac{e-1}{2e^3} < B_2 < \dfrac{e-1}{e^3}$, $\dfrac{e-1}{2e^4} < B_3 < \dfrac{e-1}{e^4}$

(4) $\dfrac{e-1}{2e^{n+1}} < B_n < \dfrac{e-1}{e^{n+1}}$

Part III Solution

(h) (1) $A_1 + B_1 = \dfrac{e-1}{e^2}$

$B_1 = \dfrac{e-1}{e^2} - A_1$

(2) $A_2 + B_2 = \dfrac{e-1}{e^3}$

$B_2 = \dfrac{e-1}{e^3} - A_2$

(3) $A_n + B_n = \dfrac{e-1}{e^{n+1}}$

$B_n = \dfrac{e-1}{e^{n+1}} - A_n$

(i) (1) Concave upward

(2) The area of A_n is always less than the area of B_n.

(j) (1) 0

(2) $A_n + B_n = \dfrac{e-1}{e^{n+1}}$

$0 < B_n < A_n < A_n + B_n$

$\therefore 0 < B_n < A_n < \dfrac{e-1}{e^{n+1}}$

$\lim_{n\to\infty} 0 = 0$

$\lim_{n\to\infty} \dfrac{e-1}{e^{n+1}} = \lim_{n\to\infty} \dfrac{e}{e^{n+1}} - \lim_{n\to\infty} \dfrac{1}{e^{n+1}}$

$\lim_{n\to\infty} \dfrac{e-1}{e^{n+1}} = \lim_{n\to\infty} \dfrac{1}{e^{n}} - \lim_{n+1\to\infty} \dfrac{1}{e^{n+1}}$

$\lim_{n\to\infty} \dfrac{e-1}{e^{n+1}} = 0 - 0$

$\lim_{n\to\infty} \dfrac{e-1}{e^{n+1}} = 0$

$\therefore \lim_{n\to\infty} A_n = \lim_{n\to\infty} B_n = 0$

Part IV Solution

(k) (1) $\frac{e-1}{2e^2} < B_1 < \frac{e-1}{e^2}$

$\frac{e-1}{2e^3} < B_2 < \frac{e-1}{e^3}$

$\frac{e-1}{2e^4} < B_3 < \frac{e-1}{e^4}$

$\cdots$

$\frac{e-1}{2e^{n+1}} < B_n < \frac{e-1}{e^{n+1}}$

$\therefore \frac{e-1}{2e^2} + \frac{e-1}{2e^3} + \frac{e-1}{2e^4} + \cdots + \frac{e-1}{2e^{n+1}}$

$< B_1 + B_2 + B_3 + \cdots + B_n < \frac{e-1}{e^2} + \frac{e-1}{e^3} + \frac{e-1}{e^4} + \cdots + \frac{e-1}{e^{n+1}}$

$\frac{1}{2}\left(\frac{e-1}{e^2} + \frac{e-1}{e^3} + \frac{e-1}{e^4} + \cdots + \frac{e-1}{e^{n+1}}\right) < \sum_{k=1}^{n} B_k < \frac{e-1}{e^2} + \frac{e-1}{e^3} + \frac{e-1}{e^4} + \cdots + \frac{e-1}{e^{n+1}}$

$\frac{1}{2}\left(\frac{1}{e} - \frac{1}{e^2} + \frac{1}{e^2} - \frac{1}{e^3} + \frac{1}{e^3} - \frac{1}{e^4} + \cdots + \frac{1}{e^n} - \frac{1}{e^{n+1}}\right)$

$< \sum_{k=1}^{n} B_k < \frac{1}{e} - \frac{1}{e^2} + \frac{1}{e^2} - \frac{1}{e^3} + \frac{1}{e^3} - \frac{1}{e^4} + \cdots + \frac{1}{e^n} - \frac{1}{e^{n+1}}$

$\frac{1}{2}\left(\frac{1}{e} - \frac{1}{e^{n+1}}\right) < \sum_{k=1}^{n} B_k < \frac{1}{e} - \frac{1}{e^{n+1}}$

$\frac{1}{2}\left(\frac{e^n}{e^{n+1}} - \frac{1}{e^{n+1}}\right) < \sum_{k=1}^{n} B_k < \frac{e^n}{e^{n+1}} - \frac{1}{e^{n+1}}$

$\frac{e^n - 1}{2e^{n+1}} < \sum_{k=1}^{n} B_k < \frac{e^n - 1}{e^{n+1}}$

(2) $\lim_{n\to\infty} \frac{e^n - 1}{e^{n+1}} = \lim_{n\to\infty}\left(\frac{1}{e} - \frac{1}{e^{n+1}}\right)$

$\lim_{n\to\infty} \frac{e^n - 1}{e^{n+1}} = \frac{1}{e} - 0$

$\lim_{n\to\infty} \frac{e^n - 1}{e^{n+1}} = \frac{1}{e}$

(3) $$\lim_{n\to\infty}\frac{e^n-1}{2e^{n+1}}<\lim_{n\to\infty}\sum_{k=1}^{n}B_k<\lim_{n\to\infty}\frac{e^n-1}{e^{n+1}}$$

$$\frac{1}{2}\lim_{n\to\infty}\frac{e^n-1}{e^{n+1}}<\lim_{n\to\infty}\sum_{k=1}^{n}B_k<\lim_{n\to\infty}\frac{e^n-1}{e^{n+1}}$$

$$\frac{1}{2}\left(\frac{1}{e}\right)<\lim_{n\to\infty}\sum_{k=1}^{n}B_k<\frac{1}{e}$$

$$\frac{1}{2e}<\lim_{n\to\infty}\sum_{k=1}^{n}B_k<\frac{1}{e}$$

(l) (1) $\frac{1}{e}-\frac{1}{e^2}+\frac{1}{e^3}-\frac{1}{e^4}+\frac{1}{e^5}-\frac{1}{e^6}+\cdots$ is the sum to infinity of the geometric sequence with the first term $\frac{1}{e}$ and the common ratio $-\frac{1}{e}$.

$$\frac{1}{e}-\frac{1}{e^2}+\frac{1}{e^3}-\frac{1}{e^4}+\frac{1}{e^5}-\frac{1}{e^6}+\cdots=\frac{\frac{1}{e}}{1-\left(-\frac{1}{e}\right)}$$

$$\frac{1}{e}-\frac{1}{e^2}+\frac{1}{e^3}-\frac{1}{e^4}+\frac{1}{e^5}-\frac{1}{e^6}+\cdots=\frac{\frac{1}{e}}{1+\frac{1}{e}}$$

$$\frac{1}{e}-\frac{1}{e^2}+\frac{1}{e^3}-\frac{1}{e^4}+\frac{1}{e^5}-\frac{1}{e^6}+\cdots=\frac{1}{e+1}$$

(2) $$\frac{e-1}{2e^2}<B_1<\frac{e-1}{e^2}$$

$$\frac{e-1}{2e^4}<B_3<\frac{e-1}{e^4}$$

$$\frac{e-1}{2e^6}<B_5<\frac{e-1}{e^6}$$

$\cdots$

$$\frac{e-1}{2e^{2n}}<B_{2n-1}<\frac{e-1}{e^{2n}}$$

$\cdots$

$$\therefore \frac{e-1}{2e^2}+\frac{e-1}{2e^4}+\frac{e-1}{2e^6}+\cdots+\frac{e-1}{2e^{2n}}+\cdots$$

$$<B_1+B_3+B_5+\cdots+B_{2n-1}+\cdots<\frac{e-1}{e^2}+\frac{e-1}{e^4}+\frac{e-1}{e^6}+\cdots+\frac{e-1}{e^{2n}}+\cdots$$

$$\frac{1}{2}\left(\frac{e-1}{e^2}+\frac{e-1}{e^4}+\frac{e-1}{e^6}+\cdots+\frac{e-1}{e^{2n}}+\cdots\right)$$

$$< B_1+B_3+B_5+\cdots+B_{2n-1}+\cdots < \frac{e-1}{e^2}+\frac{e-1}{e^4}+\frac{e-1}{e^6}+\cdots+\frac{e-1}{e^{2n}}+\cdots$$

$$\frac{1}{2}\left(\frac{1}{e}-\frac{1}{e^2}+\frac{1}{e^3}-\frac{1}{e^4}+\frac{1}{e^5}-\frac{1}{e^6}+\cdots+\frac{1}{e^{2n-1}}-\frac{1}{e^{2n}}+\cdots\right)$$

$$< B_1+B_3+B_5+\cdots+B_{2n-1}+\cdots$$

$$< \frac{1}{e}-\frac{1}{e^2}+\frac{1}{e^3}-\frac{1}{e^4}+\frac{1}{e^5}-\frac{1}{e^6}+\cdots+\frac{1}{e^{2n-1}}-\frac{1}{e^{2n}}+\cdots$$

$$\frac{1}{2}\left(\frac{1}{e+1}\right) < B_1+B_3+B_5+\cdots+B_{2n-1}+\cdots < \frac{1}{e+1}$$

$$\frac{1}{2(e+1)} < B_1+B_3+B_5+\cdots+B_{2n-1}+\cdots < \frac{1}{e+1}$$

(m) (1)

$$B_k = (1)(e^{-k}) - \int_k^{k+1} e^{-x}\,\mathrm{d}x$$

$$B_k = e^{-k} - \left[-e^{-x}\right]_k^{k+1}$$

$$B_k = e^{-k} - (-e^{-(k+1)} - (-e^{-k}))$$

$$B_k = e^{-k} + e^{-k-1} - e^{-k}$$

$$B_k = e^{-k-1}$$

(2)

$$B_k = e^{-k-1}$$

$$\frac{\mathrm{d}B_k}{\mathrm{d}k} = (e^{-k-1})(-1)$$

$$\frac{\mathrm{d}B_k}{\mathrm{d}k} = -e^{-k-1}$$

$$\frac{\mathrm{d}B_k}{\mathrm{d}k} < 0 \text{ for } k \geq 1$$

Thus, B_k is decreasing.

Exercise 81

1. (a) (1) $I(0)=\int_0^1 \frac{1}{\sqrt{1-x^2}}\mathrm{d}x$

$I(0)=\left[\arcsin x\right]_0^1$ A1

$I(0)=\arcsin 1-\arcsin 0$

$I(0)=\frac{\pi}{2}-0$

$I(0)=\frac{\pi}{2}$ A1

(2) $I(1)=\int_0^1 \frac{x}{\sqrt{1-x^2}}\mathrm{d}x$

Let $u=1-x^2$. M1

$\frac{\mathrm{d}u}{\mathrm{d}x}=-2x \Rightarrow -\frac{1}{2}\mathrm{d}u=x\mathrm{d}x$

$x=1\Rightarrow u=1-1^2=0$

$x=0\Rightarrow u=1-0^2=1$

$\therefore I(1)=\int_1^0 -\frac{1}{2}\cdot\frac{1}{\sqrt{u}}\mathrm{d}u$

$I(1)=-\frac{1}{2}\left[2\sqrt{u}\right]_1^0$ A1

$I(1)=-\frac{1}{2}(0-2)$

$I(1)=1$ A1

(3) $I(n)=\int_0^1 \frac{x^n}{\sqrt{1-x^2}}\,\mathrm{d}x$

Let $\theta=\sqrt{1-x^2}$. M1

$\frac{\mathrm{d}\theta}{\mathrm{d}x}=\frac{1}{2\sqrt{1-x^2}}(-2x)=-\frac{x}{\sqrt{1-x^2}}$

$\Rightarrow -\frac{1}{x}\mathrm{d}\theta=\frac{1}{\sqrt{1-x^2}}\mathrm{d}x$

$\therefore I(n)=\int_0^1 -\frac{x^n}{x}\mathrm{d}(\sqrt{1-x^2})$ A1

$I(n)=\int_0^1 -x^{n-1}\mathrm{d}(\sqrt{1-x^2})$

$I(n)=\left[-x^{n-1}\sqrt{1-x^2}\right]_0^1-\int_0^1\sqrt{1-x^2}\mathrm{d}(-x^{n-1})$ A1

$I(n)=(-1^{n-1}\sqrt{1-1^2})-(-0^{n-1}\sqrt{1-0^2})$

$-\int_0^1\sqrt{1-x^2}\mathrm{d}(-x^{n-1})$

$I(n)=-\int_0^1\sqrt{1-x^2}\cdot-(n-1)x^{n-2}\mathrm{d}x$

$I(n)=(n-1)\int_0^1\sqrt{1-x^2}\cdot x^{n-2}\mathrm{d}x$ A1

$I(n)=(n-1)\int_0^1\frac{(1-x^2)x^{n-2}}{\sqrt{1-x^2}}\mathrm{d}x$ M1

$I(n)=(n-1)\int_0^1\frac{x^{n-2}-x^n}{\sqrt{1-x^2}}\mathrm{d}x$

$I(n)=(n-1)\left(\int_0^1\frac{x^{n-2}}{\sqrt{1-x^2}}\mathrm{d}x-\int_0^1\frac{x^n}{\sqrt{1-x^2}}\mathrm{d}x\right)$

$I(n)=(n-1)(I(n-2)-I(n))$ A1

$I(n)=(n-1)I(n-2)-(n-1)I(n)$

$nI(n)=(n-1)I(n-2)$

$I(n)=\frac{n-1}{n}I(n-2)$ AG

(4) $I(3)=\frac{3-1}{3}I(1)$

$I(3)=\frac{2}{3}(1)$

$I(3)=\frac{2}{3}$ A1

[12]

(b) (1) $H(0)=\int_0^1 \frac{1}{(x^2+1)\sqrt{1-x^2}}\,\mathrm{d}x$

Let $x=\tan\theta$. M1

$\frac{\mathrm{d}x}{\mathrm{d}\theta}=\sec^2\theta \Rightarrow \mathrm{d}x=\sec^2\theta\,\mathrm{d}\theta$

$x=1 \Rightarrow \tan\theta=1 \Rightarrow \theta=\frac{\pi}{4}$

$x=0 \Rightarrow \tan\theta=0 \Rightarrow \theta=0$

$\therefore H(0)=\int_0^{\frac{\pi}{4}} \frac{\sec^2\theta}{(\tan^2\theta+1)\sqrt{1-\tan^2\theta}}\,\mathrm{d}\theta$ A1

$H(0)=\int_0^{\frac{\pi}{4}} \frac{\sec^2\theta}{\sec^2\theta\sqrt{1-\tan^2\theta}}\,\mathrm{d}\theta$ A1

$H(0)=\int_0^{\frac{\pi}{4}} \frac{1}{\sqrt{1-\tan^2\theta}}\,\mathrm{d}\theta$

$H(0)=\int_0^{\frac{\pi}{4}} \frac{\cos\theta}{\sqrt{\cos^2\theta-\sin^2\theta}}\,\mathrm{d}\theta$ M1

$H(0)=\int_0^{\frac{\pi}{4}} \frac{\cos\theta}{\sqrt{1-\sin^2\theta-\sin^2\theta}}\,\mathrm{d}\theta$ A1

$H(0)=\int_0^{\frac{\pi}{4}} \frac{\cos\theta}{\sqrt{1-2\sin^2\theta}}\,\mathrm{d}\theta$ AG

(2) $H(0)=\int_0^{\frac{\pi}{4}} \frac{\cos\theta}{\sqrt{1-2\sin^2\theta}}\,\mathrm{d}\theta$

Let $v=\sqrt{2}\sin\theta$. M1

$\frac{\mathrm{d}v}{\mathrm{d}\theta}=\sqrt{2}\cos\theta \Rightarrow \frac{1}{\sqrt{2}}\mathrm{d}v=\cos\theta\,\mathrm{d}\theta$

$\theta=\frac{\pi}{4} \Rightarrow v=\sqrt{2}\sin\frac{\pi}{4}=1$

$\theta=0 \Rightarrow v=\sqrt{2}\sin 0=0$

$\therefore H(0)=\int_0^1 \frac{1}{\sqrt{2}\cdot\sqrt{1-v^2}}\,\mathrm{d}v$ A1

$H(0)=\frac{1}{\sqrt{2}}\int_0^1 \frac{1}{\sqrt{1-v^2}}\,\mathrm{d}v$

$H(0)=\frac{\sqrt{2}}{2}I(0)$ AG

[7]

(c) (1) $H(2)=\int_0^1 \frac{x^2}{(x^2+1)\sqrt{1-x^2}}\,\mathrm{d}x$

$H(2)=\int_0^1 \frac{x^2+1-1}{(x^2+1)\sqrt{1-x^2}}\,\mathrm{d}x$ M1

$H(2)=\int_0^1 \frac{x^2+1}{(x^2+1)\sqrt{1-x^2}}\,\mathrm{d}x$

$-\int_0^1 \frac{1}{(x^2+1)\sqrt{1-x^2}}\,\mathrm{d}x$

$H(2)=\int_0^1 \frac{1}{\sqrt{1-x^2}}\,\mathrm{d}x-\int_0^1 \frac{1}{(x^2+1)\sqrt{1-x^2}}\,\mathrm{d}x$

$H(2)=I(0)-H(0)$ A1

(2) $H(3)=\int_0^1 \frac{x^3}{(x^2+1)\sqrt{1-x^2}}\,\mathrm{d}x$

$H(3)=\int_0^1 \frac{x^3+x-x}{(x^2+1)\sqrt{1-x^2}}\,\mathrm{d}x$ M1

$H(3)=\int_0^1 \frac{x(x^2+1)}{(x^2+1)\sqrt{1-x^2}}\,\mathrm{d}x$

$-\int_0^1 \frac{x}{(x^2+1)\sqrt{1-x^2}}\,\mathrm{d}x$

$H(3)=\int_0^1 \frac{x}{\sqrt{1-x^2}}\,\mathrm{d}x-\int_0^1 \frac{x}{(x^2+1)\sqrt{1-x^2}}\,\mathrm{d}x$

$H(3)=I(1)-H(1)$ A1

(3) $H(n)=I(n-2)-H(n-2)$ A1

[5]

(d) $H(4) = I(2) - H(2)$ M1

$H(4) = I(2) - (I(0) - H(0))$

$H(4) = \frac{2-1}{2} I(0) - I(0) + H(0)$ M1

$H(4) = -\frac{1}{2} I(0) + H(0)$

$H(4) = -\frac{1}{2} I(0) + \frac{\sqrt{2}}{2} I(0)$ M1

$H(4) = -\frac{1}{2}\left(\frac{\pi}{2}\right) + \frac{\sqrt{2}}{2}\left(\frac{\pi}{2}\right)$

$H(4) = \frac{\pi(\sqrt{2}-1)}{4}$ A1

[4]

2. (a) $z^n = 1$

$z^n = \cos 0 + \mathrm{i}\sin 0$ A1

$z = \cos\left(\dfrac{0+2k\pi}{n}\right) + \mathrm{i}\sin\left(\dfrac{0+2k\pi}{n}\right)$ M1

$(k = 0, 1, 2, \cdots, n-1)$

$z = \cos\dfrac{2k\pi}{n} + \mathrm{i}\sin\dfrac{2k\pi}{n}$

Thus, $\arg(\omega_k) = \dfrac{2k\pi}{n}$. AG

[2]

(b) (1) $\sin \mathrm{O\hat{P}_0P_1} = \dfrac{d_3}{\mathrm{OP_0}}$ M1

$\sin\left(\dfrac{\pi}{2} - \dfrac{\frac{2\pi}{3}}{2}\right) = \dfrac{d_3}{1}$ A1

$\cos\dfrac{\pi}{3} = \dfrac{d_3}{1}$

$d_3 = \dfrac{1}{2}$ A1

(2) $\sin \mathrm{O\hat{P}_0P_1} = \dfrac{d_4}{\mathrm{OP_0}}$ M1

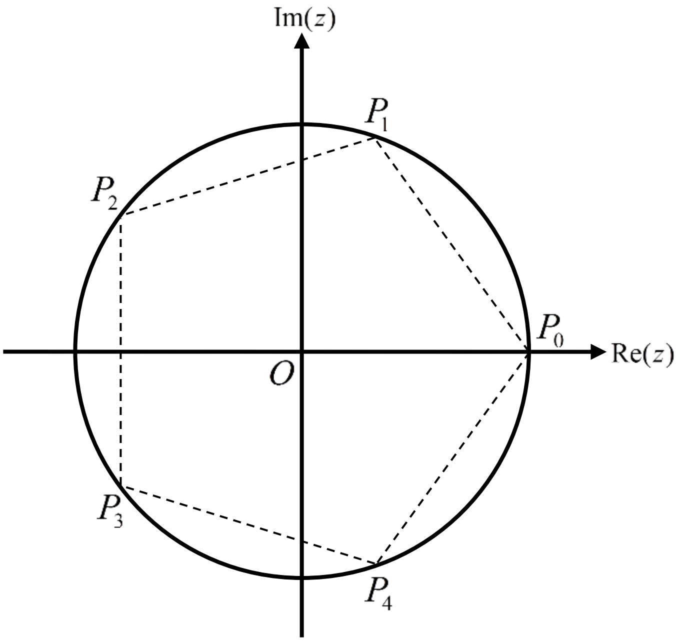
$\sin\left(\dfrac{\pi}{2} - \dfrac{\frac{2\pi}{4}}{2}\right) = \dfrac{d_4}{1}$ A1

$\cos\dfrac{\pi}{4} = \dfrac{d_4}{1}$

$d_4 = \dfrac{\sqrt{2}}{2}$ AG

(3) A1 for regular pentagon

A1 for correct arguments for P_i



(4) $d_5 = \cos\frac{\pi}{5}$ A1

(5) $\sin \text{O}\hat{\text{P}}_0\text{P}_1 = \frac{d_n}{\text{OP}_0}$ M1

$\sin\left(\frac{\pi}{2} - \frac{\frac{2\pi}{n}}{2}\right) = \frac{d_n}{1}$ A1

$\cos\frac{\pi}{n} = \frac{d_n}{1}$

$d_n = \cos\frac{\pi}{n}$ AG

[10]

(c) (1) $A_3 = 3\left(\frac{1}{2}(\text{OP}_0)(\text{OP}_1)\sin \text{P}_0\hat{\text{O}}\text{P}_1\right)$ M1

$A_3 = 3\left(\frac{1}{2}(1)(1)\sin\frac{2\pi}{3}\right)$ A1

$A_3 = 3\left(\frac{1}{2}\right)\left(\frac{\sqrt{3}}{2}\right)$

$A_3 = \frac{3\sqrt{3}}{4}$ AG

(2) $A_4 = 4\left(\frac{1}{2}(\text{OP}_0)(\text{OP}_1)\sin \text{P}_0\hat{\text{O}}\text{P}_1\right)$ M1

$A_4 = 4\left(\frac{1}{2}(1)(1)\sin\frac{2\pi}{4}\right)$ A1

$A_4 = 4\left(\frac{1}{2}\right)(1)$

$A_4 = 2$ A1

(3) $A_n = n\left(\frac{1}{2}(\text{OP}_0)(\text{OP}_1)\sin \text{P}_0\hat{\text{O}}\text{P}_1\right)$ M1

$A_n = n\left(\frac{1}{2}(1)(1)\sin\frac{2\pi}{n}\right)$ A1

$A_n = \frac{n}{2}\sin\frac{2\pi}{n}$ AG

[7]

(d) $A_n = \frac{n}{2}\sin\frac{2\pi}{n}$

$A_n = \frac{n}{2}\left(2\sin\frac{\pi}{n}\cos\frac{\pi}{n}\right)$ A1

$A_n = n\left(\sqrt{1-\cos^2\frac{\pi}{n}}\right)\cos\frac{\pi}{n}$

$A_n = nd_n(\sqrt{1-d_n^{\ 2}})$ A1

[2]

(e) (1) $\lim_{n\to\infty} n\sqrt{1-d_n^{\ 2}} = \lim_{n\to\infty} n\sqrt{1-\cos^2\frac{\pi}{n}}$

$\lim_{n\to\infty} n\sqrt{1-d_n^{\ 2}} = \lim_{n\to\infty} n\sin\frac{\pi}{n}$ A1

$\lim_{n\to\infty} n\sqrt{1-d_n^{\ 2}} = \lim_{n\to\infty} \frac{\sin\frac{\pi}{n}}{\frac{1}{n}}$

$\lim_{n\to\infty} n\sqrt{1-d_n^{\ 2}} = \lim_{n\to\infty} \frac{\left(\cos\frac{\pi}{n}\right)\left(-\frac{\pi}{n^2}\right)}{-\frac{1}{n^2}} \left(\because \frac{0}{0}\right)$ M1A1

$\lim_{n\to\infty} n\sqrt{1-d_n^{\ 2}} = \lim_{n\to\infty} \pi\cos\frac{\pi}{n}$

$\lim_{n\to\infty} n\sqrt{1-d_n^{\ 2}} = \pi\cos 0$

$\lim_{n\to\infty} n\sqrt{1-d_n^{\ 2}} = \pi$ AG

(2) $\lim_{n\to\infty} A_n = \lim_{n\to\infty} nd_n(\sqrt{1-d_n^{\ 2}})$

$\lim_{n\to\infty} A_n = \left(\lim_{n\to\infty} d_n\right)\left(\lim_{n\to\infty} n\sqrt{1-d_n^{\ 2}}\right)$

$\lim_{n\to\infty} A_n = \left(\lim_{n\to\infty} \cos\frac{\pi}{n}\right)\left(\lim_{n\to\infty} n\sqrt{1-d_n^{\ 2}}\right)$ M1

$\lim_{n\to\infty} A_n = (\cos 0)(\pi)$

$\lim_{n\to\infty} A_n = \pi$ A1

(3) The inscribed polygon becomes a unit circle and the shortest distance of the boundary of the polygon to the origin becomes the radius 1 as n tends to positive infinity. R1

[6]